Glaciological Modeling with the Finite Element Package Elmer/Ice

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Projects and Partners

Top-level Research Initiative
Contents

- General overview on ice-flow modeling
  - Basic equations (Stokes equations)
  - And their approximations (SIA, SSA)
- Introduction of Elmer/Ice
  - Ice sheet modelling with Elmer/Ice
- Some numerical concepts
  - Saddle point problem
  - Parallel runs
The Big Picture

- Until recently no ice sheet model (ISM) capable of correctly dealing with the **marine ice sheet** problem; tightly linked: **calving**
- Currently no deeper understanding of **basal sliding** processes
- Influence of **anisotropy** in ice flow
- ISM integration in **Earth System Models** (ESM)
  - Glaciers in warming climates
  - Local interaction of weather with ice surfaces
Ice Flow Dynamics

Small Reynolds/Froude number limit → Stokes

\[ \nabla \cdot \tau - \nabla p + \rho g = 0, \]

sum of all surface forces  gravity

Incompressible fluid (if firn is neglected)

\[ \nabla \cdot \mathbf{u} = 0 \]

sum of all volume fluxes

– Definition strain rate tensor \( \dot{\varepsilon} = \frac{1}{2} \left( \nabla \mathbf{u} + (\nabla \mathbf{u})^T \right) \)

– And invariants \( I_\dot{\varepsilon} = \text{tr}(\dot{\varepsilon}) = \nabla \cdot \mathbf{u}, \quad II_\dot{\varepsilon} = \text{tr}(\dot{\varepsilon}^2) \)
Ice Flow Dynamics

Shear thinning fluid $\rightarrow$ Norton-Hoff law with $n=3$ (Glen’s flow law)

$$\tau = \eta(II \dot{\varepsilon}) \dot{\varepsilon} = \left( \frac{A^{-1/n}(T, p)}{2} \left( \sqrt{II \dot{\varepsilon}} \right)^{(1-n)/n} \right) \dot{\varepsilon}$$

Arrhenius law for rate factor
$\rightarrow$ thermo-mechanical coupling

$$A(T, p) = 2E \exp \left( -\frac{Q}{RT'} \right)$$
Ice Flow Dynamics

Evolution of the free surface/flow height:

\[
\frac{\partial h}{\partial t} + u_x \frac{\partial h}{\partial x} + u_y \frac{\partial h}{\partial y} - u_z = a
\]

- **convective transport**
- **local change**
- **net accumulation**
- **coupling to climate**

Ice flow modeling/dynamics
Ice Flow Dynamics

\[ \frac{\partial h}{\partial t} + u_x \frac{\partial h}{\partial x} + u_y \frac{\partial h}{\partial y} - u_z = a \]
Ice Flow Dynamics

Accounting for all components of the stress tensor = Full Stokes (FS) model

“Cheating” by neglecting certain components:
- Approximations, usually based on flatness assumption
Approximations to FS

- Shallow Ice Approximation (SIA)
  \[ \epsilon = \frac{H}{L} \rightarrow 0 \]

- Only vertical shear stresses

- Continuity equation → elliptic

- Momentum balance → simple relation (PDE)
Approximations to FS

- Ice sheets and even glaciers are much longer than thick; generally SIA holds

- Limits of SIA and SSA:
  - Everywhere where $\varepsilon \to 0$ does not hold
Full Stokes?

- Stokes flow in 3D: 4 DOFs (u,v,w,p)
- Expensive!
- Greenland ice sheet (GIS): 1.7x10^6 km^2
- 500m resolution → 6x10^6 horizontal cells
- 15 vertical layers → 1x10^8 bulk elements/nodes
Elmer/Ice

Elmer (= multi-physics package) with additional routines for Glaciology
Maintained and supported by CSC
Open Source (GPL2 or later)
  – Transparence (you co-own the code)
  – Sustainability (no license fees)
  – Viral effect of GPL (new code also GPL)
Large international user community
  – Knowhow of well-established institutions
Good level of support/documentation
http://elmerice.elmerfem.org
Elmer/Ice

Finite element method (FEM)
- Able to produce adaptive meshes
- Using linear elements and standard Galerkin with Stabilized Finite Elements

Flow law $\rightarrow$ viscosity changes by order of magnitudes $\rightarrow$ bad conditioned system:
- Direct parallel Solver
- Special pre-conditioning

(Massive) parallel computing
- MPI
- Multi-threading under development
Elmer/Ice

Implemented Glaciological models:
- Unstructured meshing
- Deforming and moving meshes
- Rheology: Glen, anisotropy, firn densification
- Special sliding laws
- NetCDF-readers (for geometry as well as coupling to climate)
- Simple SMB (PDD)
- Simple calving model
- Inverse methods for data assimilation
- Methods for tracer transport/dating
Currently developed/planned Glaciological models:

- Coupling of SIA and FS
- Hydrological model for soft-bedded (till) glaciers
- Hydrological model for hard-bedded glacier
- Enhanced calving laws (including issues with mesh updates)
- Coupling to ocean model
- Till deformation model
Ice-sheets

- Challenging by size
- Centuries rather than millennia
- Needs high performance computing (HPC) facilities
Different Applications of Elmer/Ice
Full Stokes modelling
Sliding

Different types of sliding surfaces
- Till (deforming sediment)
- Hard rock

Strong involvement of bed hydrology

Bedrock processes not really understood and hard to determine from observations

**Inverse Problem**: Determine sliding conditions from surface velocities
Sliding

Solving iteratively Dirichlet (prescribed surface velocities) and Neuman (vanishing surface stress) problem

Optimization with respect to the bed friction coefficient $\beta$ by Gâteaux derivative $d_\beta J$ of cost function:

$$ J = \int_A (\bar{u}_N - \bar{u}_D) \cdot (\sigma_N - \sigma_D) \cdot \bar{n} \, dA $$
Sliding

velocities m/yr

Model

Data
Sliding
Marine Ice Sheets

- Ice sheet that terminates at sea
- Ice that passes the grounding line (GL) contributes to sea level rise (SLR)

Pattyn & Naruse (2003) J. Glacio., 49, 166
Marine Ice Sheets

Vast increase of outlet and retreat → SLR

Howat et al. (2007)
Marine Ice Sheets

- Sudden change of stress field at GL

- Characteristics of a boundary layer
Marine Ice Sheets

(Mesh) Size matters

Different steady states depending on resolution

Consistent for

\[ \Delta x < 200 \text{ m} \]

Marine Ice Sheets
Moving to 3D – pinning point
L. Favier, O. Gagliardini, G. Durand, and T. Zwinger  *A three-dimensional full Stokes model of the grounding line dynamics: effect of a pinning point beneath the ice shelf*  
The Cryosphere Discuss., 5, 1995-2033, 2011
Calving

Columbia glacier, Alaska, USA

Nye formulation

Including water level inside crevasses

– Shift of terminus by 200 m – well within the range of observations

Anisotropy

- Mono-crystalline ice is extremely anisotropic
- Polar ice is a polycrystalline material
Anisotropy

- Fabric is influenced by shear history = dynamics
- Feedback via rheology
Material ice

Influence of anisotropy

Glaciers

Katabatic windfront impinging on blue ice area at Scharffenberg-bottnen, DML, EAAIS Elmer, VMS turbulentce model

Simulation by T. Zwinger and T. Malm
FEM Formulation

Conservation of linear momentum

\[ \nabla \cdot \tau - \nabla p + \rho g = 0, \]

\( \nabla \cdot \tau \) sum of all surface forces

\( \nabla p \) gravity

Conservation of mass

\[ \nabla \cdot \mathbf{u} = 0 \]

\( \nabla \cdot \mathbf{u} \) sum of all volume fluxes
FEM formulation

- Weak formulation:

\[- \int_{\Omega} (\mathbf{\tau} - 1p) \cdot \nabla \varphi_\alpha \, d\Omega \quad + \quad \int_{\Gamma} \mathbf{t} \varphi_\alpha \, d\Gamma \]

\[= \int_{\Omega} \varphi_\alpha \, d\Omega \]

- Linearization of deviatoric stress:

\[\mathbf{\tau}\big|_{(n)} = \eta(\dot{\mathbf{\varepsilon}}\big|_{(n-1)}) \dot{\mathbf{\varepsilon}}\big|_{(n)}\]
FEM formulation

\[ p(x) = p_1 \varphi_1 |_x + p_2 \varphi_2 |_x + p_3 \varphi_3 |_x \]
FEM formulation

- Discretization by weighting functions:

\[ u = \sum_{\beta} u_{\beta} \psi_{\beta}, \quad p = \sum_{\beta} p_{\beta} \chi_{\beta} \]

\[ p_{\beta} \int_{\Omega} \chi_{\beta} \nabla \varphi_{\alpha} d\Omega - u_{\beta} \int_{\Omega} \eta \varepsilon (\nabla \psi_{\beta}) \nabla \varphi_{\alpha} d\Omega \]

\[ + \int_{\Gamma} t \varphi_{\alpha} d\Gamma = \int_{\Omega} \rho g \varphi_{\alpha} d\Omega \]
FEM formulation

- Standard Galerkin:

$$ p_\beta \int_\Omega \varphi_\beta \nabla \varphi_\alpha d\Omega - u_\beta \int_\Omega \eta D(\nabla \varphi_\beta) \nabla \varphi_\alpha d\Omega + \int_\Gamma t \varphi_\alpha d\Gamma = \int_\Omega \omega g \varphi_\alpha d\Omega $$

$$ \psi_\beta = \chi_\beta = \varphi_\beta $$
FEM formulation

Saddle point problem: \[ S = \begin{bmatrix} -\eta D & \nabla \\ \nabla^T & 0 \end{bmatrix} = \begin{bmatrix} C^{-1} & A \\ A^T & 0 \end{bmatrix} \]

\[ \begin{bmatrix} C^{-1} & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix} \]

\( S \) is not positive definite

\( \iff \) no minimization problem

\( \iff \exists \) non-trivial null-space for \( p \) (checkerboard)

\( \iff \) Babuska-Brezzi (aka. inf-sup) condition
FEM formulation

Condition for **stability:**

\[
\inf_p \sup_u \frac{p^T A u}{(u^T C^{-1} u) (p^T p)} \geq C
\]

- Depends on the space of the test-functions
- **Stabilization.** Methods in Elmer:
  1. Residual square methods
  2. Residual free bubbles
  3. Taylor-Hood
  4. VMS

\[
\begin{bmatrix}
C^{-1} & A \\
A^T & B
\end{bmatrix}
\cdot
\begin{bmatrix}
u \\
p
\end{bmatrix}
=
\begin{bmatrix}f \\
0
\end{bmatrix}
\]
Contact Problem

- No-penetration condition at bedrock
- Solving: \( \frac{\partial h}{\partial t} + u_x \frac{\partial h}{\partial x} + u_y \frac{\partial h}{\partial y} - u_z = a_\perp, \)

in combination with: \( h > b + \Delta h_{\min} \)

- Variational formulation \(\rightarrow\) Variational inequality

Courtesy Lionel Favier, LGGE
Contact Problem

Solving system:
\[ S_{ij} h_j = a_i, \quad h_j > h_{j,\text{min}} \]

Flag, , for active nodes initialized with 0

Marking nodes with:
\[ h_j < h_{j,\text{min}} \Rightarrow f_j = 1 \]

Manipulations:
\[ \forall f_j = 1 : S_{ij} \rightarrow S'_{ij} = \delta_{ij}, \quad a_i = h_{j,\text{min}} \]

Solving modified system:
\[ S'_{ij} h_j = a'_i \]

Residual of unaltered system:
\[ R_i = S_{ij} h_j - a_i \]

Flag turned back to 0:
\[ h_j > h_{j,\text{min}} \land R_i < 0 \]

Consistent, fast and robust algorithm
Contact Problem
FEM Formulation

Different types of elements:

- Standard Lagrangian
- Second order (edge + face centered)
- P-elements (highest order: 9)
- Mixed type elements (Taylor-Hood, Mini-element)
- Discontinuous Galerkin

BEM solver (e.g., Accoustics)

Lagrangian particle tracker (alternative to DG)
Linear Algebra

Internal CRS (compact row storage) scheme

Different solution methods in Elmer:

1. Direct solution: Unsymmetrical Multi-Frontal method UMFPack (serial) and MUMPS (parallel)
2. Iterative = Krylov subspace: GMRES, CG, BiCGStab, GCR
3. Multi-Grid: AMG (built-in), Boomer AMG

Pre-conditioner:
- Diagonal, ILU(N/T)
- Hypre: Parasails, BoomerAMG
- Block pre-conditioner
Block pre-conditioning
Linear Algebra

Not re-inventing the wheel → interfacing with comming linear algebra packages:

– LAPack
– UMFPACK
– MUMPS (ScaLAPack)
– Hypre (pre-conditioners and AMG)
– Pardiso (multi-threaded library)
– SparsIter (Cholmod)
– Trilinos (J. Thies)
– To be continued
Parallel Concept of Elmer

Modern HPC CPU’s already have 6+ cores
- Core frequency stagnating
- Need to go parallel for improvements

Full Stokes is expensive
- 4 DOFs (3 x momentum, pressure)

Greenland Ice Sheet:
- $1.7 \times 10^6$ km$^2$
- 500 m resolution $\rightarrow$ $6 \times 10^6$ horizontal cells
Parallel Concept of Elmer

- Domain decomposition
- Additional pre-processing step (splitting)
- Every domain is running its "own" ElmerSolver
- Parallel process communication: Message Passing Interface (MPI)
- Re-combination of ElmerPost output
Parallel Concept of Elmer

unpartitioned mesh → partitioned mesh

domain decomposition

parallel solution

unification of result

combined result
Parallel Concept of Elmer

Scaling of wall clock time with dofs in the cavity lid case using GMRES+ILU0.
Simulation Juha Ruokolainen, CSC, visualization Matti Gröhn, CSC.
Parallel Concept of Elmer

- Altered numerics in parallel
- "Missing" parts of global system matrix, e.g., in ILU
- Communication expensive
- Direct parallel solver (MUMPS)
  - Memory hog
  - Does not scale at all
Thank you!