Laboratoire de Glaciologie et Géophysique de l'Environnement

Elmer/Ice advanced Workshop

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Lower-order Stokes model

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Outline

- **Shallow Shelf / Shallow stream Solver**
- **Thickness Solver**
- **Shallow Ice Solver**
- **Current / planned development**

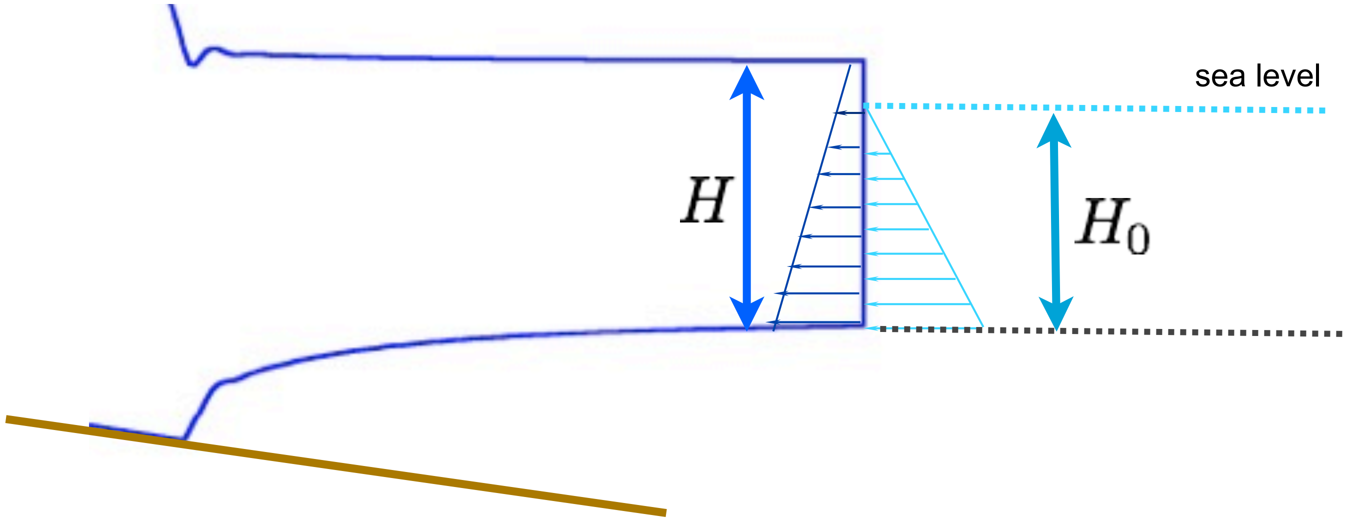
Shallow Shelf Approximation/Shallow Stream Approximation

Field equations:

$$\begin{cases} \frac{\partial}{\partial x}\left(2H\nu\left(2\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right)+\frac{\partial}{\partial y}\left(H\nu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right)-\beta u=\rho g H\frac{\partial z_s}{\partial x} \\ \frac{\partial}{\partial x}\left(H\nu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right)+\frac{\partial}{\partial y}\left(2H\nu\left(\frac{\partial u}{\partial x}+2\frac{\partial v}{\partial y}\right)\right)-\beta v=\rho_i g H\frac{\partial z_s}{\partial y} \end{cases}$$

Boundary Conditions:

$$\begin{cases} 4H\nu\frac{\partial u}{\partial x}n_x+2H\nu\frac{\partial v}{\partial y}n_x+H\nu(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial x})n_y=(\rho_i gH-\rho_w gH_0)n_x \\ 4H\nu\frac{\partial v}{\partial y}n_y+2H\nu\frac{\partial v}{\partial x}n_y+H\nu(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial x})n_x=(\rho_i gH-\rho_w gH_0)n_y \end{cases}$$



Shallow Shelf Approximation/Shallow Stream Approximation

Field equations:

$$\begin{cases} \dfrac{\partial}{\partial x}\left(2H\nu\left(2\dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}\right)\right)+\dfrac{\partial}{\partial y}\left(H\nu\left(\dfrac{\partial v}{\partial x}+\dfrac{\partial u}{\partial y}\right)\right)-\beta u=\rho g H\dfrac{\partial z_s}{\partial x} \\ \dfrac{\partial}{\partial x}\left(H\nu\left(\dfrac{\partial v}{\partial x}+\dfrac{\partial u}{\partial y}\right)\right)+\dfrac{\partial}{\partial y}\left(2H\nu\left(\dfrac{\partial u}{\partial x}+2\dfrac{\partial v}{\partial y}\right)\right)-\beta v=\rho_i g H\dfrac{\partial z_s}{\partial y} \end{cases}$$

$$H = Zs - Zb$$

Elmer/Ice Solvers:

Solver Fortran File: `SSASolver.f90`
Solver Name: `SSABasalSolver`

`Required Output Variable(s):`

- `SSAVelocity`

Required Input Variable(s):

- (1) `Zb`, `Zs` and `Effective Pressure` when using the Coulomb type friction law

The `SSABasalSolver` solve the classical SSA equation, it has been modified in Rev. 6440 to be executed either on a grid of dimension lower than the problem dimension itself (i.e. the top or bottom grid of a 2D or 3D mesh for a SSA 1D or 2D problem), or on a grid of the same dimension of the problem (i.e. 2D mesh for a 2D plane view SSA solution).

It will work on a 3D mesh only if the mesh as been extruded along the vertical direction and if the base line boundary conditions have been preserved (to impose neumann conditions). **Keyword *«Preserve Baseline = Logical True»* in section Simulation**

Shallow Shelf Approximation/Shallow Stream Approximation

Field equations:

$$\begin{cases} \dfrac{\partial}{\partial x}\left(2H\nu\left(2\dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}\right)\right)+\dfrac{\partial}{\partial y}\left(H\nu\left(\dfrac{\partial v}{\partial x}+\dfrac{\partial u}{\partial y}\right)\right)-\beta u=\rho g H\dfrac{\partial z_s}{\partial x} \\ \dfrac{\partial}{\partial x}\left(H\nu\left(\dfrac{\partial v}{\partial x}+\dfrac{\partial u}{\partial y}\right)\right)+\dfrac{\partial}{\partial y}\left(2H\nu\left(\dfrac{\partial u}{\partial x}+2\dfrac{\partial v}{\partial y}\right)\right)-\beta v=\rho_i g H\dfrac{\partial z_s}{\partial y} \end{cases}$$

SIF - Solver Section:

```
Solver 1
  Equation = "SSA"
  Procedure = File "ElmerIceSolvers" "SSABasalSolver"
  Variable = String "SSAVelocity"
  Variable DOFs = 2    ! 1 in SSA 1-D or 2 in SSA-2D

  Linear System Solver = Direct
  Linear System Direct Method = umfpack

  Nonlinear System Max Iterations = 100
  Nonlinear System Convergence Tolerance  = 1.0e-08
  Nonlinear System Newton After Iterations = 5
  Nonlinear System Newton After Tolerance = 1.0e-05

  Nonlinear System Relaxation Factor = 1.00

  Steady State Convergence Tolerance = Real 1.0e-3
End
```

Shallow Shelf Approximation/Shallow Stream Approximation

Field equations:

$$\begin{cases} \frac{\partial}{\partial x}\left(2H\nu\left(2\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right)+\frac{\partial}{\partial y}\left(H\nu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right)-\beta u=\rho gH\frac{\partial z_s}{\partial x} \\ \frac{\partial}{\partial x}\left(H\nu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right)+\frac{\partial}{\partial y}\left(2H\nu\left(\frac{\partial u}{\partial x}+2\frac{\partial v}{\partial y}\right)\right)-\beta v=\rho_i gH\frac{\partial z_s}{\partial y} \end{cases}$$

SIF - Material Section:

```
Material 1
! Flow Law
  Viscosity Exponent = Real $1.0/n
  Critical Shear Rate = Real 1.0e-10
  SSA Mean Viscosity = Real $eta
  SSA Mean Density = Real $rhoi

! Friction Law
  ! Which law are we using
  SSA Friction Law = String («linear», «weertman» or «coulomb»)

  ! friction parameter
  SSA Friction Parameter = Real 0.1

! Needed for Weertman and Coulomb
  ! Exponent m
  SSA Friction Exponent = Real $1.0/n

  ! Min velocity for linearisation where ub=0
  SSA Friction Linear Velocity = Real 0.0001

! Needed for Coulomb only
  ! post peak exponent in the Coulomb law (q, in Gagliardini et al., 2007)
  SSA Friction Post-Peak = Real ...
  ! Iken's bound  tau_b/N < C (see Gagliardini et al., 2007)
  SSA Friction Maximum Value = Real ....
  SSA Min Effective Pressure = Real ...
End
```

Friction laws:

- Linear:

$$\tau_b = \beta u$$

- Weertman:

$$\tau_b = \beta |u|^{(m-1)} u$$

- Coulomb:

$$\tau_b = \frac{1}{A_s^{\frac{1}{n}}}\left[\frac{1}{\left(1+\alpha . \chi^{q}\right)}\right]^{\frac{1}{n}} . u_b^{\frac{1}{n}-1} . u$$

$$\alpha = \frac{(q-1)^{q-1}}{q^{q}} \qquad \chi = \frac{u_b}{C^{n} N^{n} A_s}$$

Shallow Shelf Approximation/Shallow Stream Approximation

Field equations:

$$\begin{cases} \dfrac{\partial}{\partial x}\left(2H\nu\left(2\dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}\right)\right)+\dfrac{\partial}{\partial y}\left(H\nu\left(\dfrac{\partial v}{\partial x}+\dfrac{\partial u}{\partial y}\right)\right)-\beta u=\rho g H\dfrac{\partial z_s}{\partial x} \\ \dfrac{\partial}{\partial x}\left(H\nu\left(\dfrac{\partial v}{\partial x}+\dfrac{\partial u}{\partial y}\right)\right)+\dfrac{\partial}{\partial y}\left(2H\nu\left(\dfrac{\partial u}{\partial x}+2\dfrac{\partial v}{\partial y}\right)\right)-\beta v=\rho_i g H\dfrac{\partial z_s}{\partial y} \end{cases}$$

Boundary Conditions:

$$\begin{cases} 4H\nu\dfrac{\partial u}{\partial x}n_x+2H\nu\dfrac{\partial v}{\partial y}n_x+H\nu(\dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial x})n_y=(\rho_i g H-\rho_w g H_0)n_x \\ 4H\nu\dfrac{\partial v}{\partial y}n_y+2H\nu\dfrac{\partial v}{\partial x}n_y+H\nu(\dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial x})n_x=(\rho_i g H-\rho_w g H_0)n_y \end{cases}$$

SIF - Boundary Conditions / Constants / Body Forces:

```
Boundary Condition 1
! Dirichlet condition
  SSAVelocity 1 = Real ...
  SSAVelocity 2 = Real ...
End
Boundary Condition 1
! Neumann Condition
  Calving Front = Logical True
End
```

```
Constants
! Used for Neumann condition
  Water Density = Real ....
  Sea Level = Real ...
End
```

```
Body Force 1
! The gravity from Flow Body Force 2/3 (1D/2D)
    Flow BodyForce 3 = Real $gravity
End
```

Computing mean values

SSA uses mean viscosity and density:

$$\nu(x,y) = \frac{1}{H}\int_{z_b}^{z_s} \mu(x,y,z)dz$$

→ coupling with : **Temperature, Damage**

$$\bar{\rho}(x,y) = \frac{1}{H}\int_{z_b}^{z_s} \rho(x,y,z)dz$$

→ coupling with : **Porous solver**

You can use:

Elmer/Ice solver : *`GetMeanValueSolver`*

- **unstructured** meshes in the vertical direction

```
Solver 1
  Equation = "SSA-IntValue"
  Procedure = File "ElmerIceSolvers" "GetMeanValueSolver"
  Variable = -nooutput String "Integrated variable"
  Variable DOFs = 1

  Exported Variable 1 = String "Mean Viscosity"
  Exported Variable 1 DOFs = 1
  Exported Variable 2 = String "Mean Density"
  Exported Variable 2 DOFs = 1

  Linear System Solver = Direct
  Linear System Direct Method = umfpack

  Steady State Convergence Tolerance = Real 1.0e-3
End


!!! Upper free surface
Boundary Condition 1
  Depth = Real 0.0
  Mean Viscosity = Real 0.0
  Mean Density = real 0.0
End
```

Elmer solver : *`StructuredProjectToPlane`*

- **structured** meshes in the vertical direction

```
Solver 1
  Equation = "HeightDepth"
  Procedure = "StructuredProjectToPlane" "StructuredProjectToPlane"
  Active Coordinate = Integer 3

  Operator 1 = depth
  Operator 2 = height
  Operator 3 = thickness

  !! compute the integrated horizontal Viscosity and Density
  Variable 4 = Viscosity
  Operator 4 = int

  Variable 5 = Density
  Operator 5 = int
End

Material 1
  SSA Mean Viscosity = Variable "int Viscosity", thickness
       REAL MATC "tx(0)/tx(1)"
  SSA Mean Density = Variable "int Density", thickness
       REAL MATC "tx(0)/tx(1)"
End
```

=> J. Brondex (LGGE) is working on new solutions for this step and to compute the 3D velocity field
(=> coupling with damage and temperature)

Outline

- Shallow Shelf / Shallow stream Solver
- **Thickness Solver**
- Shallow Ice Solver
- Current / planned development

Thickness Solver

Field equations:

$$\frac{\partial H}{\partial v} + \nabla(\bar{u} H) = a_s + a_b$$

Elmer/Ice Solvers:

- **Solver Fortran File:** `ThicknessSolver.f90`
- **Solver Name:** `ThicknessSolver`
- **Required Output Variable(s):** `H`
- **Required Input Variable(s):** `H residual`
- **Optional Output Variable(s):** `dhdt`
- **Optional Input Variable(s):** `FlowSolution`

• This solver is based on the FreeSurfaceSolver and use a **SUPG stabilsation** scheme by **default** (***residual free bubble stabilization*** can be use instead).

• As for the FreeSurfaceSolver ***Min* and *Max* limiters** can be used.

• As for the Free surface solver **only a Dirichlet boundary condition** can be imposed.

• This solver can be used on a mesh of the same dimension as the problem (e.g. solve on the bottom or top boundary of a 3D mesh to solve the 2D thickness field) or on a mesh of lower dimension (e.g. can be use in a 2D plane view mesh with the SSA Solver solver for example)

Thickness Solver

Field equations:

$$\frac{\partial H}{\partial v} + \nabla(\bar{u}H) = a_s + a_b$$

SIF:

```
Solver 1
   Equation = "Thickness"
   Variable = -dofs 1 "H"

   Exported Variable 1 = -dofs 1 "H Residual"

!! To compute dh/dt
   Exported Variable 2 = -dofs 1 "dHdt"
   Compute dHdT = Logical True

  Procedure = "ElmerIceSolvers" "ThicknessSolver"
!   Before Linsolve = "EliminateDirichlet" "EliminateDirichlet"

   Linear System Solver = Direct
   Linear System Direct Method = umfpack
   Linear System Convergence Tolerance = Real 1.0e-12

! equation is linear if no min/max
   Nonlinear System Max Iterations = 50
   Nonlinear System Convergence Tolerance  = 1.0e-6
   Nonlinear System Relaxation Factor = 1.00

! stabilisation method: [stabilized\bubbles]
  Stabilization Method = stabilized

!! to apply Min/Max limiters
  Apply Dirichlet = Logical True

!! to use horizontal ALE formulation
   ALE Formulation = Logical True

!! To get the mean horizontal velocity
!!  either give the name of the variable
     Flow Solution Name = String "SSAVelocity"
!!!!! or give the dimension of the problem using:
!    Convection Dimension = Integer
End
```

```
Body Force 1
!! Mass balance
  Top Surface Accumulation = Real ....
  Bottom Surface Accumulation = Real ....


!! if the convection velocity is not directly given by a variable
!! Then give //Convection Dimension = Integer// in the solver section
!! and the Mean velocity here:
  Convection Velocity 1 = Variable int Velocity 1, thickness
     REAL MATC "tx(0)/tx(1)"
  Convection Velocity 2 = Variable int Velocity 2, thickness
     REAL MATC "tx(0)/tx(1)"

End
```

```
Boundary Condition 1
! Dirichlet condition only
  H = Real ...
End
```

```
Material 1
!! Limiters
  Min H = Real ....
  Max H = Real ....

End
```

Coupling SSA solver / Thickness solver

SSASolver uses Zs and Zb (H=Zs-Zb)

=> requires an intermediate step between *ThicknessSolver* and *SSASolver*

Do it yourself:

```
Initial Condition 1
  H = Real ....
End

Body Force 1
! to update Zb and Zs according to H evolution
  Zb = Real ...
  Zs = Variable Zb , H
     REAL MATC "tx(0)+tx(1)"
End

Solver 1
   Equation = "UpdateExport"
   Procedure = "ElmerIceSolvers" "UpdateExport"
   Variable = -nooutput "dumy"

    Exported Variable 1 = -dofs 1 "Zb"
    Exported Variable 2 = -dofs 1 "Zs"
End

Solver 2
  Equation = "SSA"
  Procedure = File "ElmerIceSolvers" "SSABasalSolver"
  Variable = String "SSAVelocity"
  Variable DOFs = 2   ! 1 in SSA 1-D
End

Solver 3
   Equation = "Thickness"
   Variable = -dofs 1 "H"
End
```

you can write a User Function to apply flotation to Zb and Zs=Zb+H

1. From H compute Zb and Zs

look for definition of Exported variables in «Body Force»

2. From Zb and Zs compute u

3. From u compute H

Coupling SSA solver / Thickness solver

SSASolver uses Zs and Zb (H=Zs-Zb)

=> requires an intermediate step between *ThicknessSolver* and *SSASolver*

Do it yourself:

```
Initial Condition 1
  H = Real ....
End

Body Force 1
! to update Zb and Zs according to H evolution
  Zb = Real ...
  Zs = Variable Zb , H
     REAL MATC "tx(0)+tx(1)"
End

Solver 1
   Equation = "UpdateExport"
   Procedure = "ElmerIceSolvers" "UpdateExport"
   Variable = -nooutput "dumy"

    Exported Variable 1 = -dofs 1 "Zb"
    Exported Variable 2 = -dofs 1 "Zs"
End

Solver 2
  Equation = "SSA"
  Procedure = File "ElmerIceSolvers" "SSABasalSolver"
  Variable = String "SSAVelocity"
  Variable DOFs = 2    ! 1 in SSA 1-D
End

Solver 3
   Equation = "Thickness"
   Variable = -dofs 1 "H"
End
```

I will put a *FlotationSolver* in the distrib soon:

- From H apply Flotation to compute Zb
- If bedrock is given check if floating or grounded
- compute grounded mask (-1: floating, +1: grounded, 0: groundig line)
- Zs = H + Zb
- *optionally:* compute dZs/dt and dZb/dt

Check volume and fluxes using SaveScalars

```
Solver X
  Exec Solver = After Timestep

  Equation = "Save Scalars"
  Procedure = File "SaveData" "SaveScalars"

  Filename = File "Scalars_"$name$".dat"

  Variable 1 = "Time"

! int H = Volume
  Variable 2 = "H"
  Operator 2 = "int"

! int dh/dt = dVolume/dt
 Variable 3 = "dhdt"
 Operator 3 = "int"

! int SMB
  Variable 4 = "smb"
  Operator 4 = "int"

! SMB_H=Artificial additionnal Mass flux due to limits on H
  Variable 5 = "h residual"
  Operator 5 = "sum"

! OUT Flow
  Variable 6 = "SSAVelocity"
  Operator 6 = "convective flux"
  Coefficient 6 = "Flux"

!=> Dvolume/dt ~ SMB + SMB_H - OUT
End
```

```
Material 1
!! For Save scalar to compute mass flux (=H*SSA_UV)
   Flux = Equals H
End

Boundary Condition 1
  Target Boundaries = 1

  Save Scalars = Logical True

  Calving Front = Logical True

End
```

Examples

Friction Laws:

ismip diagnostic test cases

[ELMER_TRUNK]/elmerice/Tests/SSA_Coulomb
[ELMER_TRUNK]/elmerice/Tests/SSA_Weertman

Coupling SSA/Thickness:

[ELMER_TRUNK]/elmerice/Tests/SSA_IceSheet

[ELMER_TRUNK]/elmerice/examples/Test_SSA → ismip prognostic test:

- 1D (2D mesh)
- 2D (2D mesh)
- 2D (3D mesh; use *StructuredProjectToPlane* to compute mean values))

Coupling SSA/Stokes:

ismip prognostic test:

*[ELMER_TRUNK]/elmerice/*Tests/ThicknessSolver

Outline

- Shallow Shelf / Shallow stream Solver
- Thickness Solver
- **Shallow Ice Solver**
- Current / planned development

Shallow Ice Approximation

Field equations:

$$
\begin{aligned}
\frac{\partial u}{\partial z} &= -2A(\rho g)^n (S-z)^n \left[\sqrt{(\frac{\partial S}{\partial x})^2 + (\frac{\partial S}{\partial y})^2}\right]^{n-1} \frac{\partial S}{\partial x}\,, \\
&= -(\rho g/\eta)^n (S-z)^n \left[\sqrt{(\frac{\partial S}{\partial x})^2 + (\frac{\partial S}{\partial y})^2}\right]^{n-1} \frac{\partial S}{\partial x}\,,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial v}{\partial z} &= -2A(\rho g)^n (S-z)^n \left[\sqrt{(\frac{\partial S}{\partial x})^2 + (\frac{\partial S}{\partial y})^2}\right]^{n-1} \frac{\partial S}{\partial y}\,, \\
&= -(\rho g/\eta)^n (S-z)^n \left[\sqrt{(\frac{\partial S}{\partial x})^2 + (\frac{\partial S}{\partial y})^2}\right]^{n-1} \frac{\partial S}{\partial y}\,,
\end{aligned}
$$

$$\frac{\partial w}{\partial z} = -\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\,,$$

$$\frac{\partial p}{\partial z} = -\rho g\,,$$

These equations can be seen as degenerated Poisson equations:

$$\frac{\partial^2 U}{\partial z^2} = \Psi(x,y,z)\,, \quad \text{with boundary conditions} \quad \frac{\partial U}{\partial z} = \Gamma(x,y) \text{ for } z = \Omega_f\,, \text{ and } U = \bar{U} \text{ for } z = \Omega_u\,. \qquad (1)$$

The SIA Solver in the *Elmer/Ice* distrib solves 4 times Eq. (1) for the 4 unkwons (u,v,w,p)
=> work with fully unstructured meshes
=> we need to work on efficient solutions for structured meshes

Shallow Ice Approximation

SIF entries:

```
! Dummy solver just here to declare SIAFlow
! as a true variable (not an exported variable)
! to allow access to previous values
Solver 2
  Equation = "SIA Variable"
  Procedure = File "ElmerIceSolvers" "SIAVariable"
  Variable = "SIAFlow"
  Variable DOFs = 4  ! 4 in 3D (u,v,w,p), 3 in 2D (u,v,p)
End

Solver 3
  Equation = "SIA"
  Procedure = File "ElmerIceSolvers" "SIASolver"
  Variable = -nooutput "SIAvar"
  Variable DOFs = 1

  Linear System Solver = "Direct"
  Linear System Direct Method = umfpack

  Steady State Convergence Tolerance = Real 1.0e-3
End

!!! bedrock
Boundary Condition 5
  Target Boundaries = 5
  SIAFlow 1 = Real 0.0e0
  SIAFlow 2 = Real 0.0e0
  SIAFlow 3 = Real 0.0e0
End

!!! free surface
Boundary Condition 6
  Target Boundaries = 6
  Save Line = Logical True
  SIAFlow 4 = real 0.0  !(p=0)
  Depth = real 0.0
End
```

An example using the SIASolver applied to experiment A160 of ISMIP-HOM benchamrks can be found in `[ELMER_TRUNK]/elmerice/Tests/SIA`.

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- Shallow Ice Solver
- **Current / planned development**

Current/planned developments

Inverse methods:

- AdjointSolver for SSA => constrain friction, mean viscosity, Zb, Zs from observation
- AdjointSolver for Thickness => constrain u,smb from observation of H

(see Morlighem *et al.*, 2011, a mass consservation approach for mapping glacier ice thickness)

SSA*:

- modify viscosity to take into account vertical shearing

(see Cornford *et al.*, 2013, adaptative mesh, finite volume modeling of marine ice sheets)

Efficient hybrid model SSA+SIA

Efficient coupling with Temperature and Damage

Current/planned developments

Adaptative mesh refinement around the grounding line:

- serial mesh-splitting strategy already implemented