A SIA Solver for Elmer/Ice

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Theory

The Shallow Ice Approximation (SIA) is well used in Glaciology. For a 3D problem, the velocities (u, v, w) and isotropic pressure p can be inferred from the 4 following equations :

$$\frac{\partial u}{\partial z} = -2A(\rho g)^n (S-z)^n \left[\sqrt{(\frac{\partial S}{\partial x})^2 + (\frac{\partial S}{\partial y})^2}\right]^{n-1} \frac{\partial S}{\partial x}\,, \tag{1}$$

$$= -(\rho g/\eta)^n (S-z)^n \left[\sqrt{(\frac{\partial S}{\partial x})^2 + (\frac{\partial S}{\partial y})^2}\right]^{n-1} \frac{\partial S}{\partial x}\,, \tag{2}$$

$$\frac{\partial v}{\partial z} = -2A(\rho g)^n (S-z)^n \left[\sqrt{(\frac{\partial S}{\partial x})^2 + (\frac{\partial S}{\partial y})^2}\right]^{n-1} \frac{\partial S}{\partial y}\,, \tag{3}$$

$$= -(\rho g/\eta)^n (S-z)^n \left[\sqrt{(\frac{\partial S}{\partial x})^2 + (\frac{\partial S}{\partial y})^2}\right]^{n-1} \frac{\partial S}{\partial y}\,, \tag{4}$$

$$\frac{\partial w}{\partial z} = -\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\,, \tag{5}$$

and

$$\frac{\partial p}{\partial z} = -\rho g\,, \tag{6}$$

in which $z = S$ denotes the surface altitude, A is the Glen's flow law parameter, η the viscosity in ELMER, $n = 1/m$ the Glen's flow law exponent with m the viscosity exponent in ELMER, ρ the ice density and g the norm of the gravity vector.

The boundary conditions for the four variables are $u(x, y, B) = u_b$, $v(x, y, B) = v_b$, $w(x, y, B) = w_b$ and $p(x, y, S) = 0$, where $z = B$ is the altitude of the bedrock.

These four equations can be solved by using the same kind of solver than the Flowdepth solver (they are also degenerated Poisson equation). We have to solve the following differential equation:

$$\frac{\partial^2 U}{\partial z^2} = \Psi(x, y, z)\,, \tag{7}$$

with the boundary conditions

$$\frac{\partial U}{\partial z} = \Gamma(x,y) \text{ for } z = \Omega_f \text{ , and } U = \bar{U} \text{ for } z = \Omega_u \, . \tag{8}$$

• For the velocity u, one can identify:

$$\Psi = n(\rho g/\eta)^n (S-z)^{n-1} \left[\sqrt{(\frac{\partial S}{\partial x})^2 + (\frac{\partial S}{\partial y})^2} \right]^{n-1} \frac{\partial S}{\partial x} \left(1 + \frac{\partial \eta}{\partial z}(S-z)/\eta \right) \, , \tag{9}$$

with $\Gamma = 0$ on $z = S$ and $u = u_b$ on $z = B$.

• For the velocity v:

$$\Psi = n(\rho g)^n (S-z)^{n-1} \left[\sqrt{(\frac{\partial S}{\partial x})^2 + (\frac{\partial S}{\partial y})^2} \right]^{n-1} \frac{\partial S}{\partial y} \left(1 + \frac{\partial \eta}{\partial z}(S-z)/\eta \right) \, , \tag{10}$$

with $\Gamma = 0$ on $z = S$ and $v = v_b$ on $z = B$.

• For the velocity w:

$$\Psi = -\frac{\partial^2 u}{\partial xz} - \frac{\partial^2 u}{\partial yz} \, , \tag{11}$$

with $\Gamma = -\left.\frac{\partial u}{\partial x}\right|_{z=S} - \left.\frac{\partial v}{\partial y}\right|_{z=S}$ on $z = S$ and $w = w_b$ on $z = B$.

• For the isotropic pressure p:

$$\Psi = 0 \, , \tag{12}$$

with $\Gamma = -\rho g$ on $z = B$ and $p = 0$ on $z = S$.

Specificities

This SIA solver is not classical in that sense that the equations are not solved on a grid of dimension lower than the problem dimension itself. The geometry (H, B and S) is here given by the mesh. For a flow line problem, the mesh is a plane surface, and a volume for a 3D problem. Regarding this aspect, this solver is certainly not as efficient as a classical SIA solver. But, on the other hand, it works for unstructured grid and non-constant viscosity. The SIA velocities and pressure can be use, for example, as initial conditions for the Navier-Stokes Solver. Contrary to the NS solver, the gravity must be orientated along the $z-$axis.

The SIA solver uses the same input parameters as the NS solver (`Viscosity`, `Density`, `Viscosity Exponent`, `Flow BodyForce`, ...).

The basal velocities are given as Dirichlet BC on the bedrock surface. The SSA Solver can be used to this purpose.