Elmer/Ice advanced Workshop

30 Nov – 2 Dec 2015

Ice Rheologies

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Rheology of Ice(s)

✓ **The Physics**
  - Ice(s) on Earth
  - Important internal variables

✓ **Rheological laws**
  - Glen's flow law
  - Anisotropic laws (GOLF and CAFFE)
  - A law for the firn/snow
  - Associated evolution equations (fabric, density)
  - Damage

✓ **Implementation in Elmer/Ice**
  - AIFlow Solver and Fabric Solver
  - Porous Solver
  - User Function USF_Damage

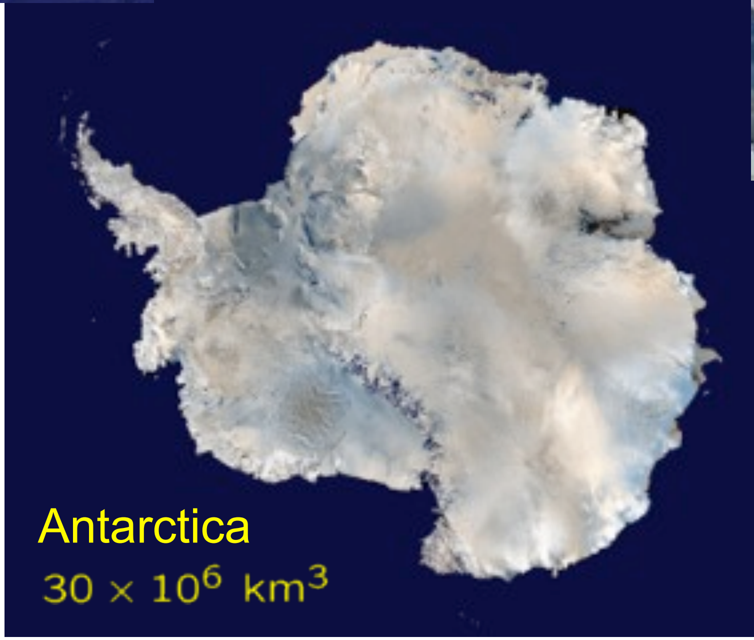
Flowing ice(s) on the Earth

Greenland

2×10^6 km^3

Glaciers (~220 000)

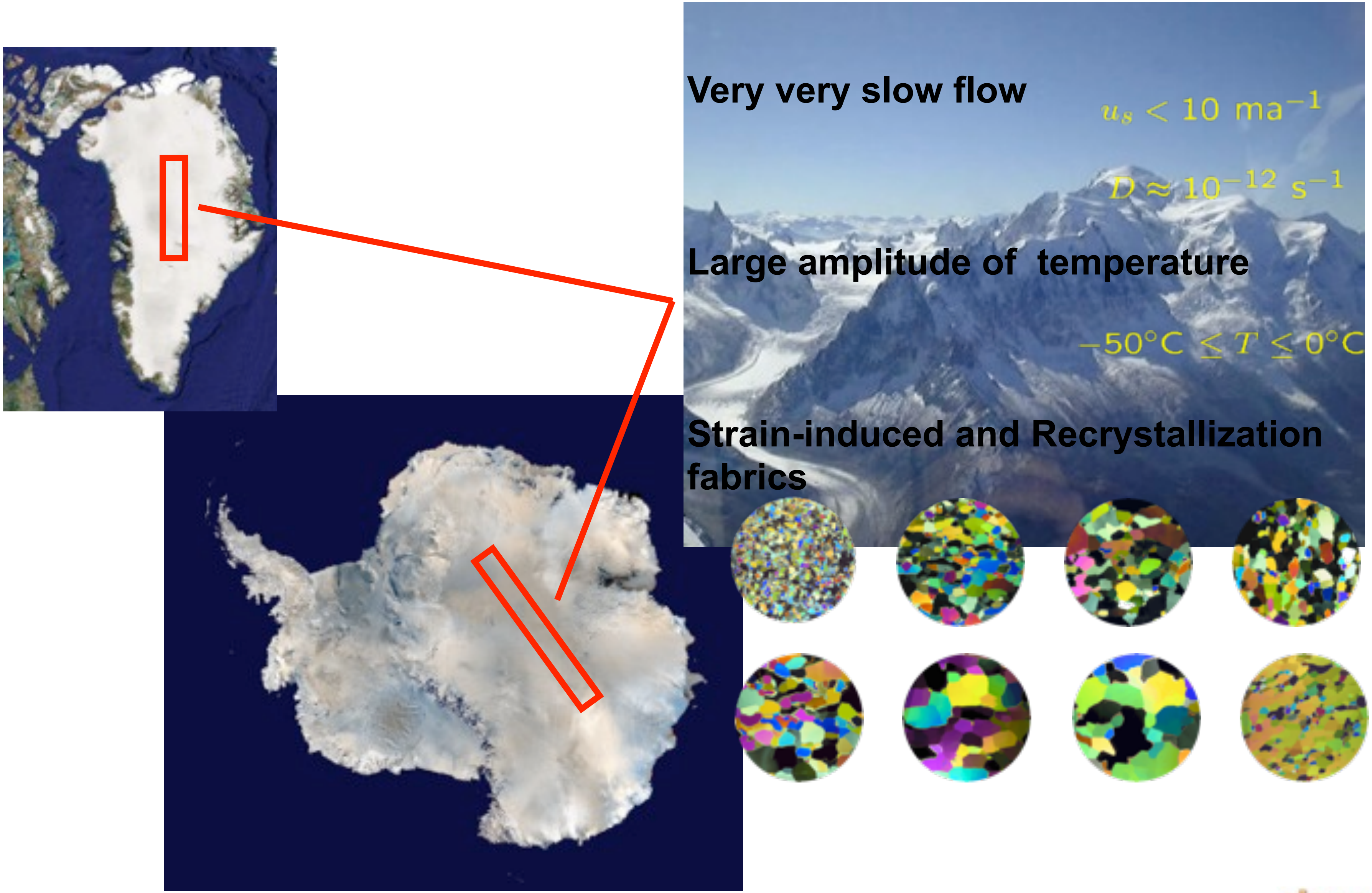
550×10^3 km^2



Antarctica

30×10^6 km^3

Central part of ice-sheets



Very very slow flow

$u_s < 10 \text{ ma}^{-1}$

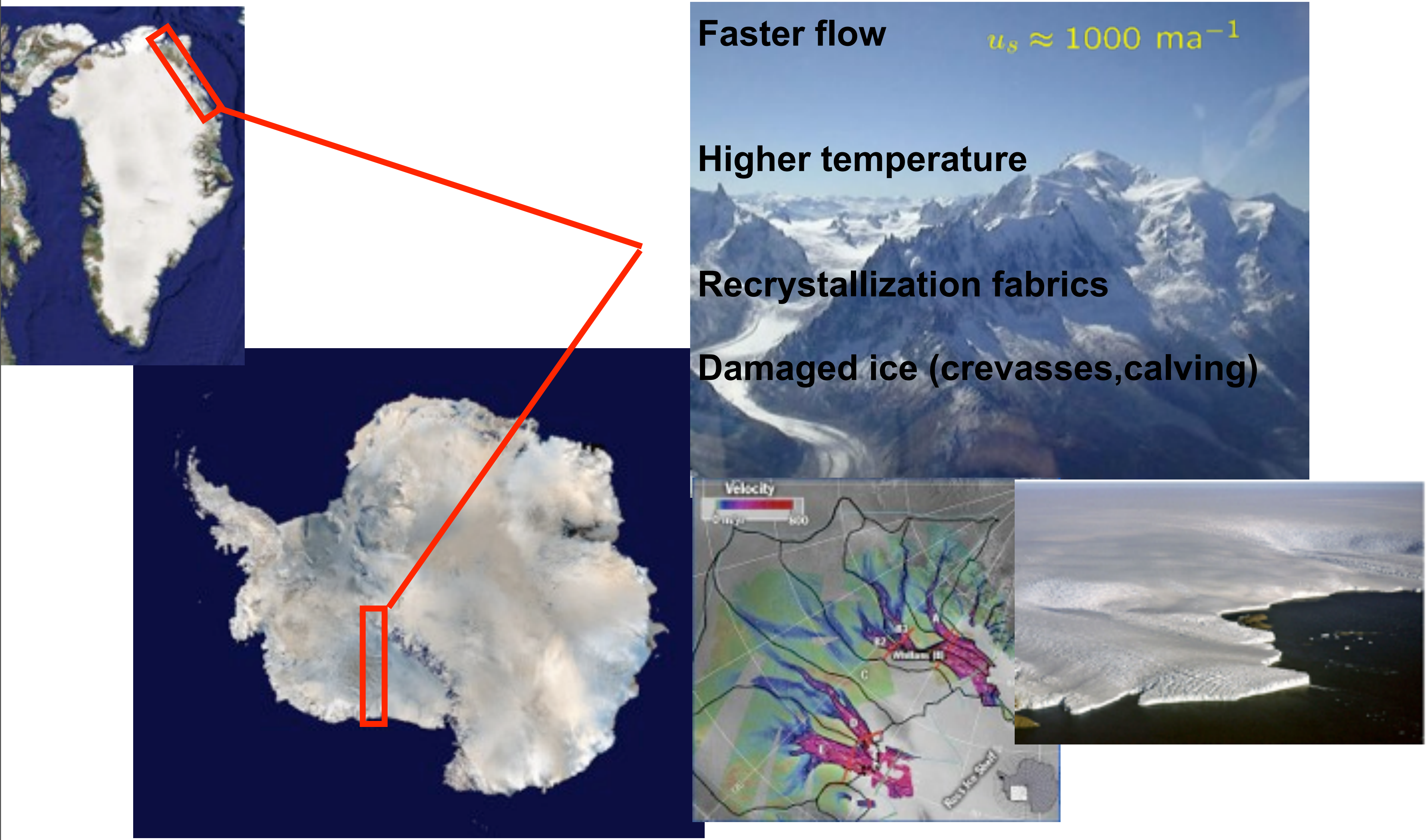
$D \approx 10^{-12} \text{ s}^{-1}$

Large amplitude of temperature

$-50°\text{C} \lesssim T \lesssim 0°\text{C}$

Strain-induced and Recrystallization fabrics

Margin of ice-sheets



Faster flow $u_s \approx 1000 \text{ ma}^{-1}$

Higher temperature

Recrystallization fabrics

Damaged ice (crevasses,calving)

Glaciers

Faster flow

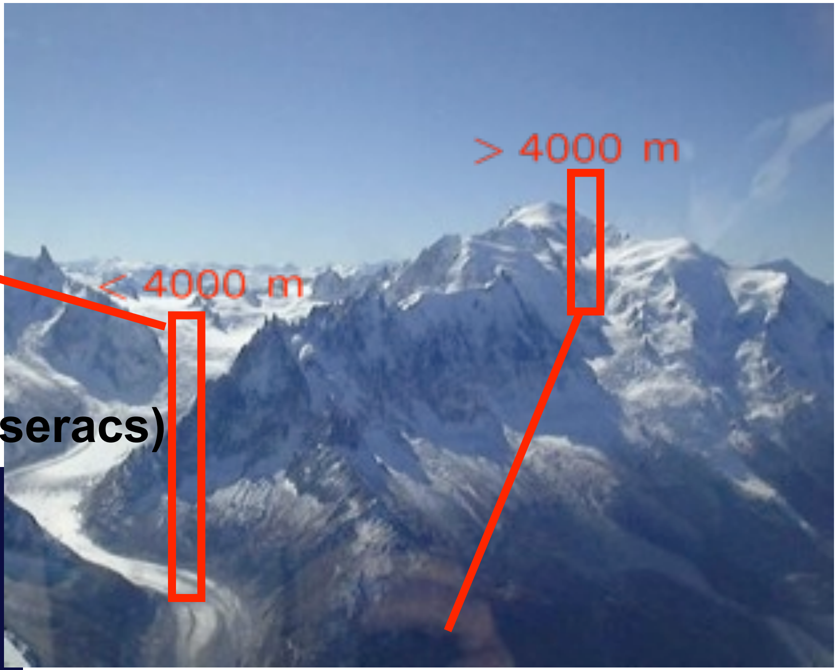
$u_s \approx 100\ \mathrm{ma}^{-1}$

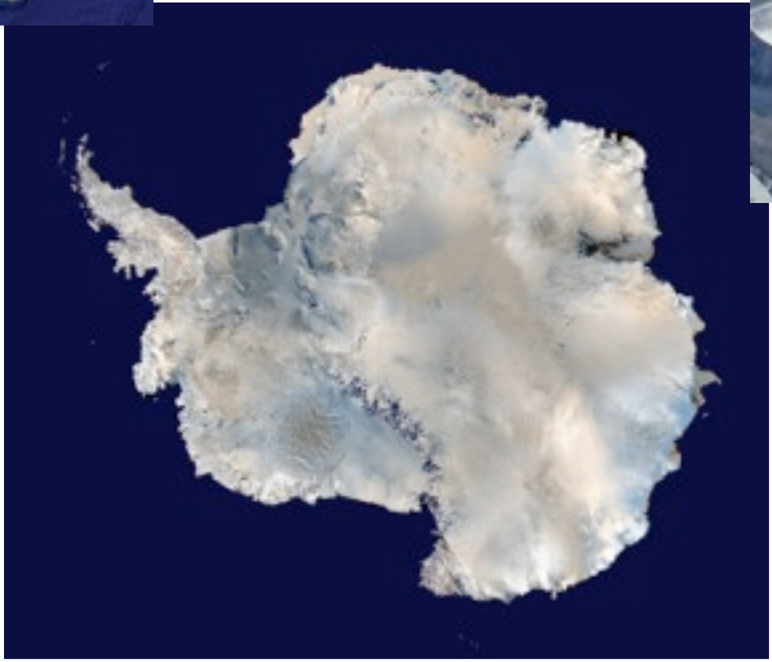
Temperate ice

$T = O^o\mathrm{C}$

Stress-induced fabrics

Damaged ice (crevasses, seracs)





Slow flow

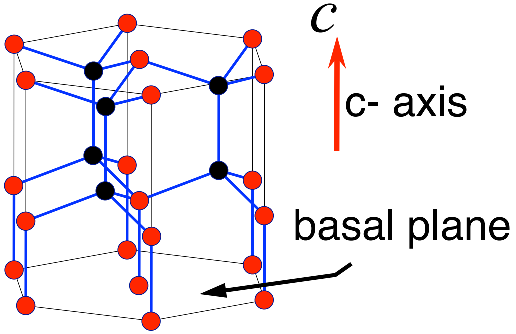
$u_s \approx 10\ \mathrm{ma}^{-1}$

Lower temperature

$T < O^o\mathrm{C}$

Large part composed by snow/firn

Ice crystal and Fabric

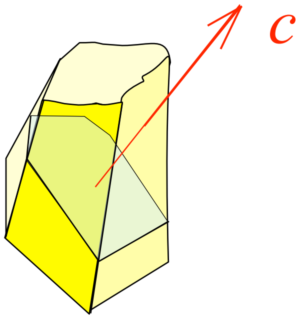


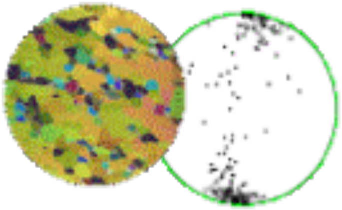
Hexagonal symmetry

One of the most anisotropic natural material

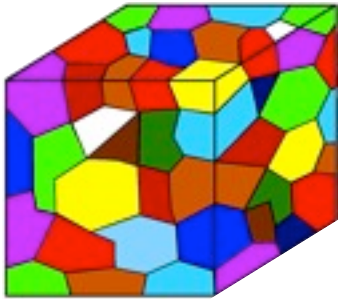
behave like a deck of cards !!

Ice crystal





Fabric

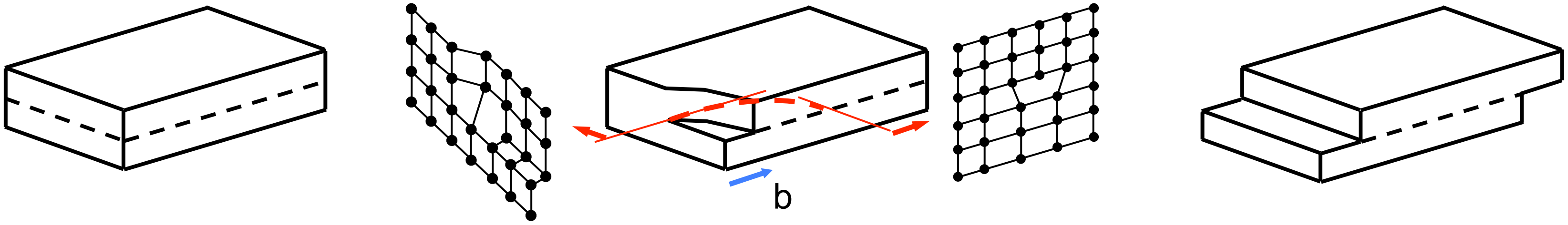


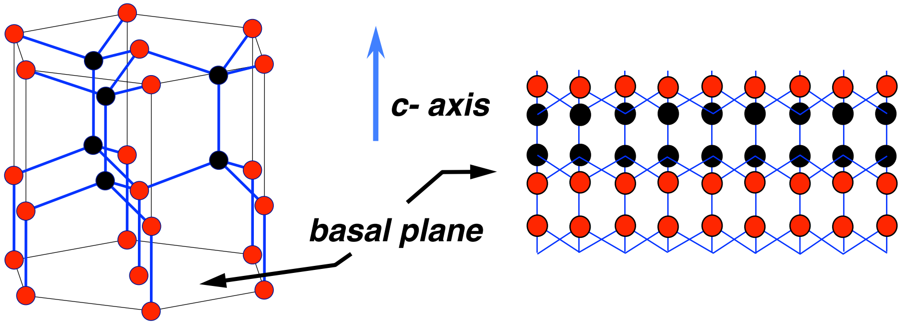
Polycrystalline ice

Anisotropy function of the fabric

Ice monocrystal viscoplastic behaviour

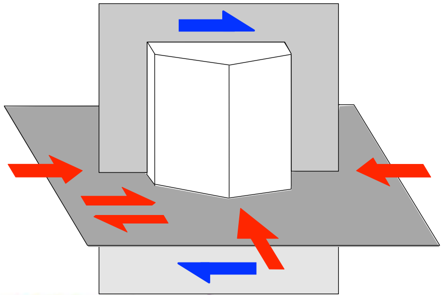
The viscoplastic deformation is due to the dislocation glide





Ice crystal deforms mainly by dislocation glide in the basal planes

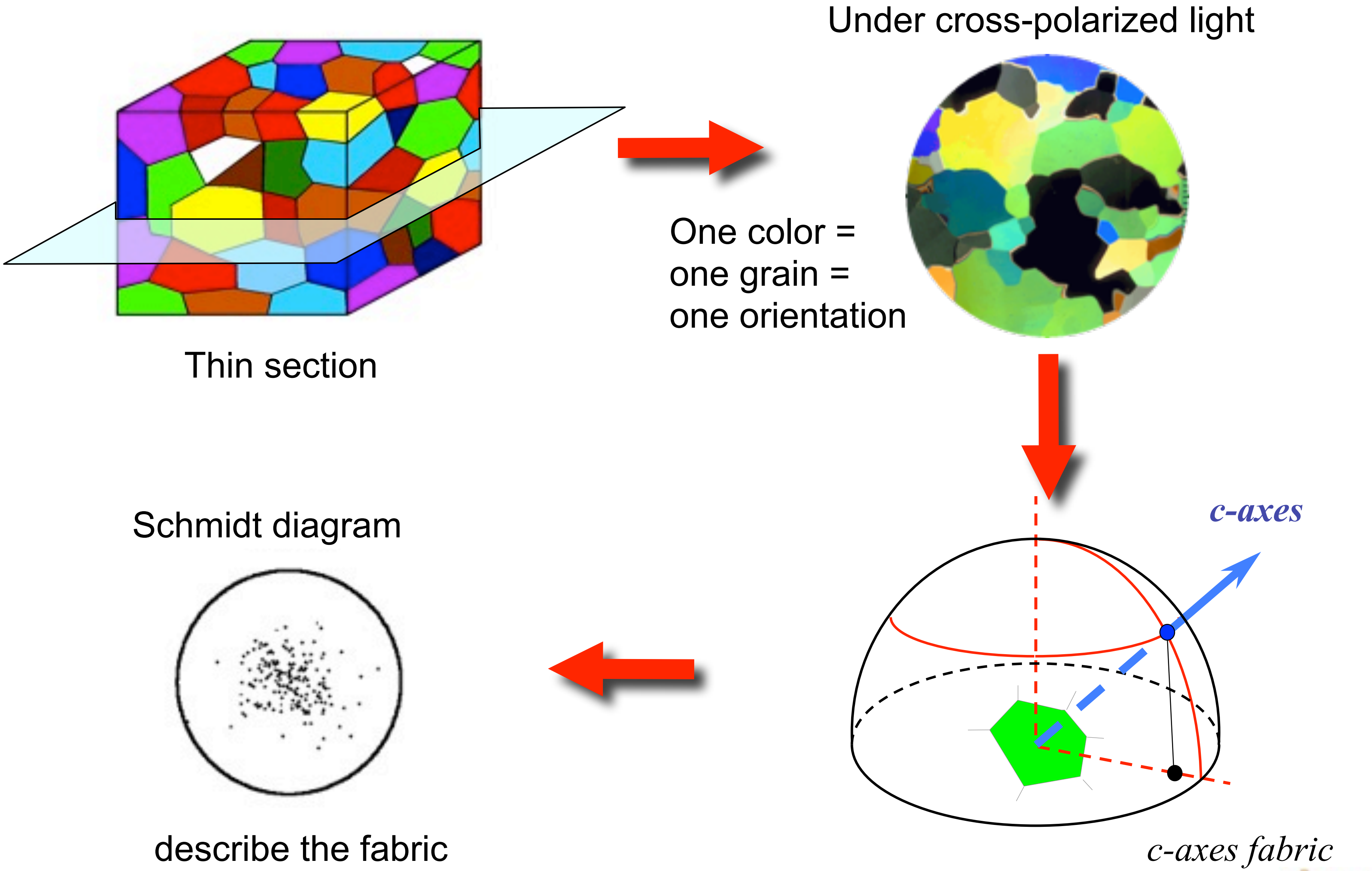
Ice is one of the most **anisotropic** natural material



Shearing parallel to basal plane is almost **1000 time faster** than compression (⊥ ou // p. b.) or shearing in the basal plane

behave like a deck of cards

Fabric of polycrystalline ice



Thin section

Under cross-polarized light

One color =
one grain =
one orientation

Schmidt diagram

describe the fabric

c-axes fabric

Observed ice fabric patterns

Depends on the strain history undergone by the polycrystal

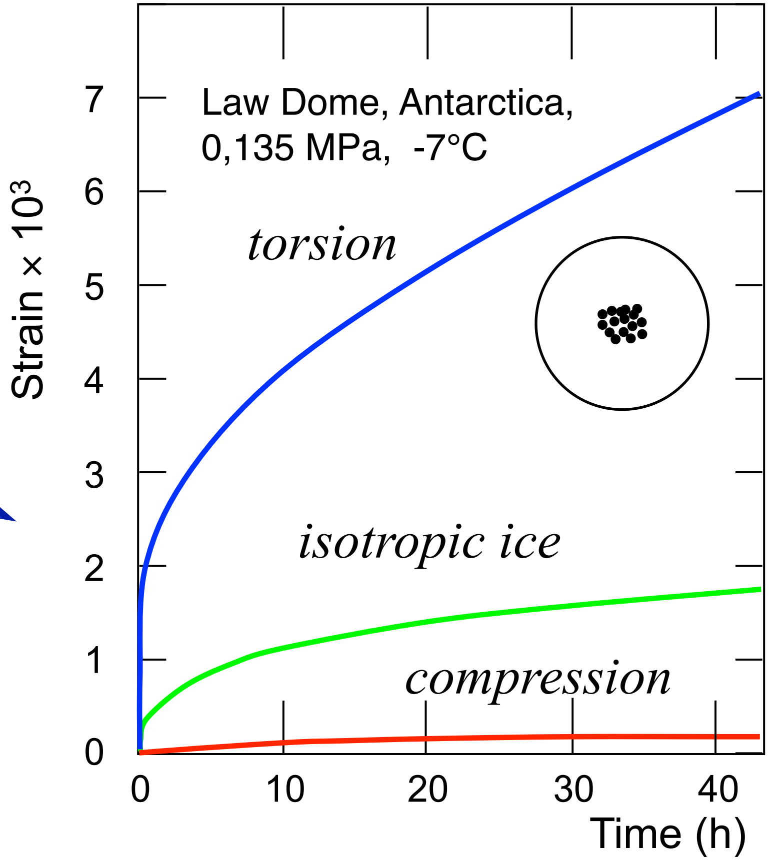
Vertical

In a vertical plane

Compression and/or
Simple shear

Tension

Anisotropic polycrystalline ice



Single maximum fabric is about 10 time easier to shear than isotropic ice

Damaged ice : a continuum mechanic approach

Objective:

- quantify the degradation of mechanical properties resulting from the nucleation of internal defects such as micro-cracks or voids

A continuum damage mechanics model:

- internal defects must be small compared to the representative volume element over which damage is considered

Damaged ice : a continuum mechanic approach

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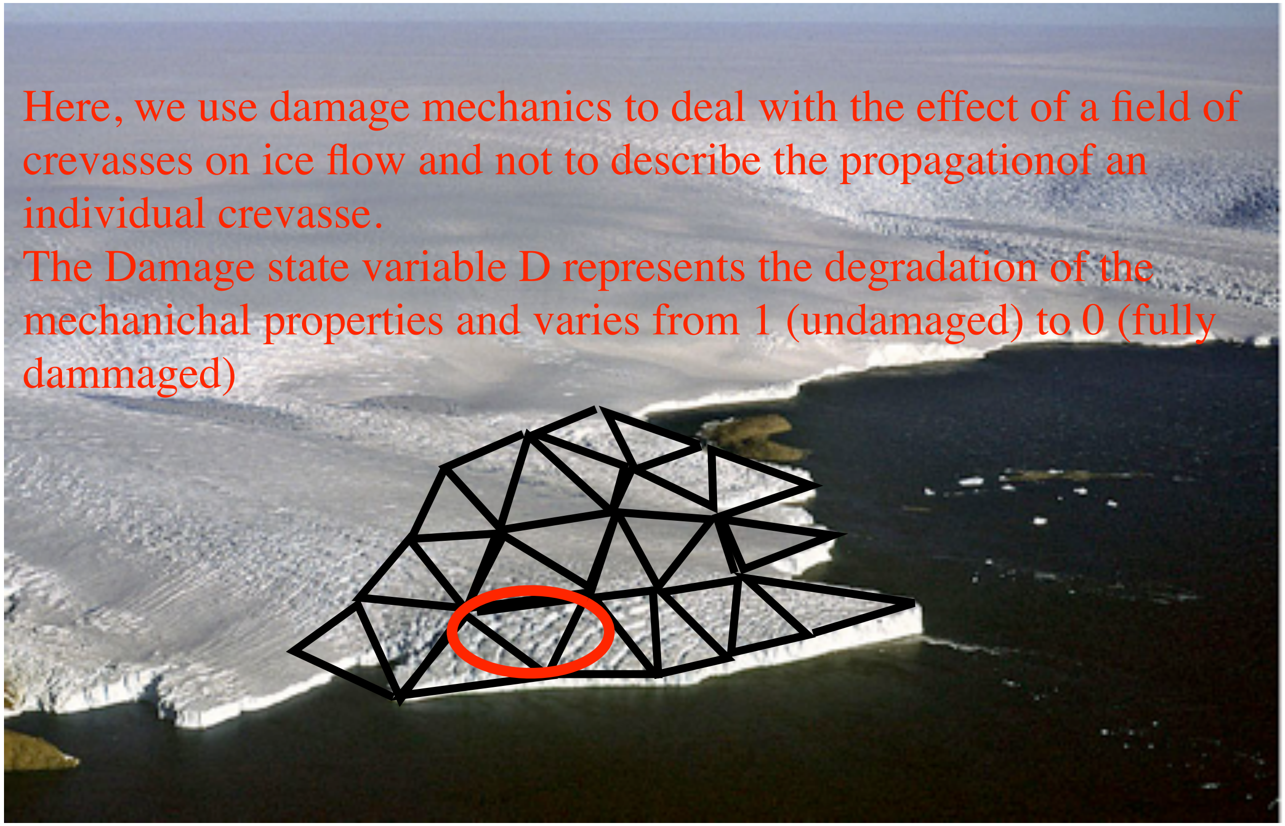
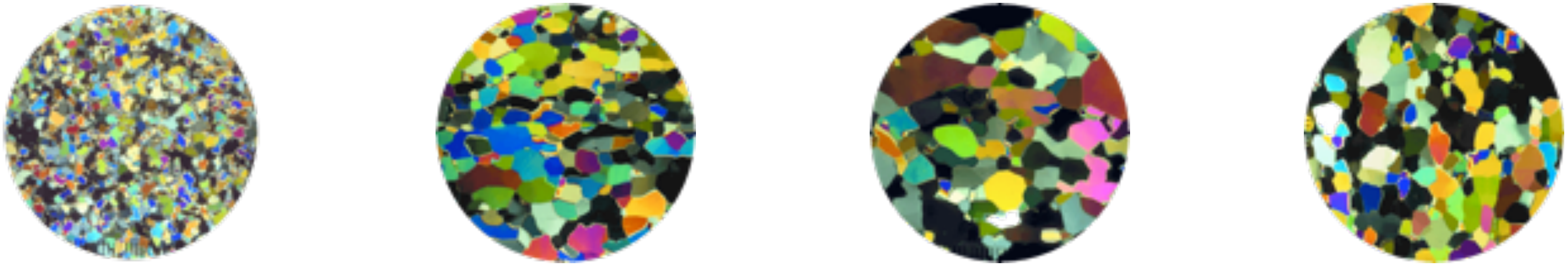


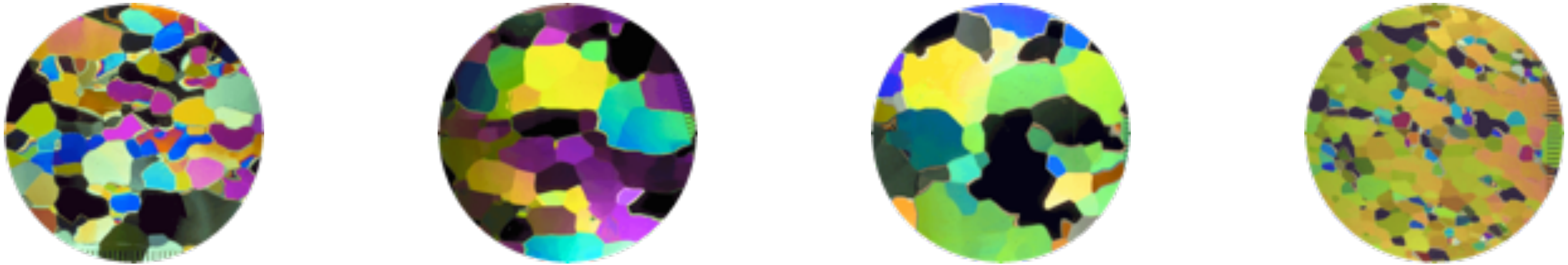
Here, we use damage mechanics to deal with the effect of a field of crevasses on ice flow and not to describe the propagationof an individual crevasse.

The Damage state variable D represents the degradation of the mechanichal properties and varies from 1 (undamaged) to 0 (fully dammaged)

Rheological properties of ice(s)



The behaviour of each piece of ice is unique !



Temperature

fabric

Size of the crystals (?)

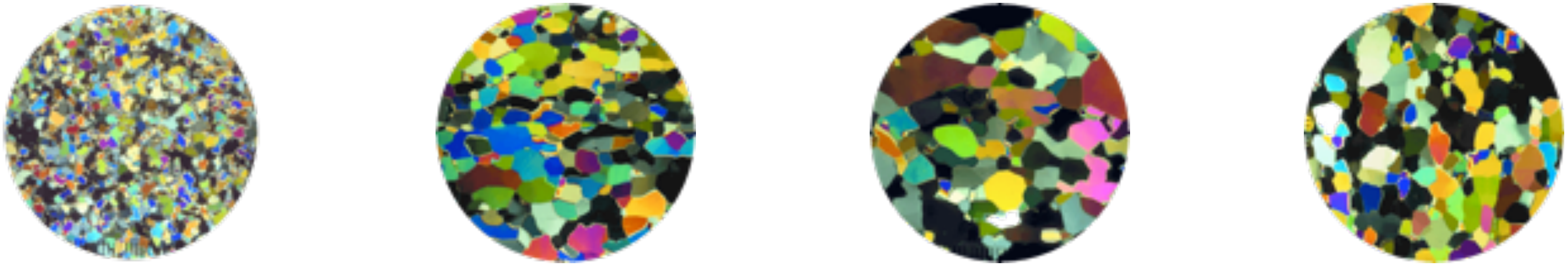
Water content

Damage

Density

Dust content

Rheological properties of ice(s)



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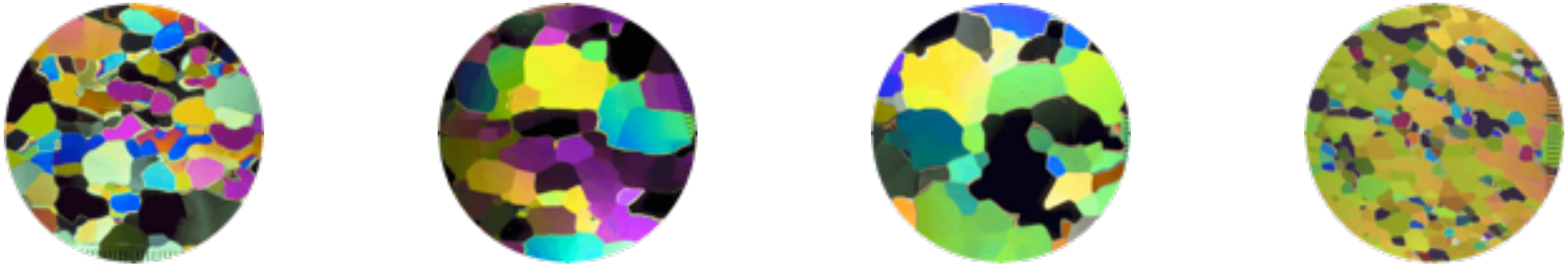

Temperature

fabric

Size of the crystals (?)

Need a law dedicated to each problem

Water content

Damage

Density

Dust content

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Isotropic Ice (Glen's law)

Isotropic ice : Norton-Hoff type law

$$D_{ij} = A\tau_e^{n-1}S_{ij} \quad ; \quad S_{ij} = A^{-1/n} I_{D_2}^{(1-n)/n} D_{ij}$$

where
$$\begin{cases} I_{D_2}^2 = D_{ij}D_{ij}/2 \\ \tau_e^2 = S_{ij}S_{ij}/2 \end{cases}$$

Arrhenius law for temperature dependency

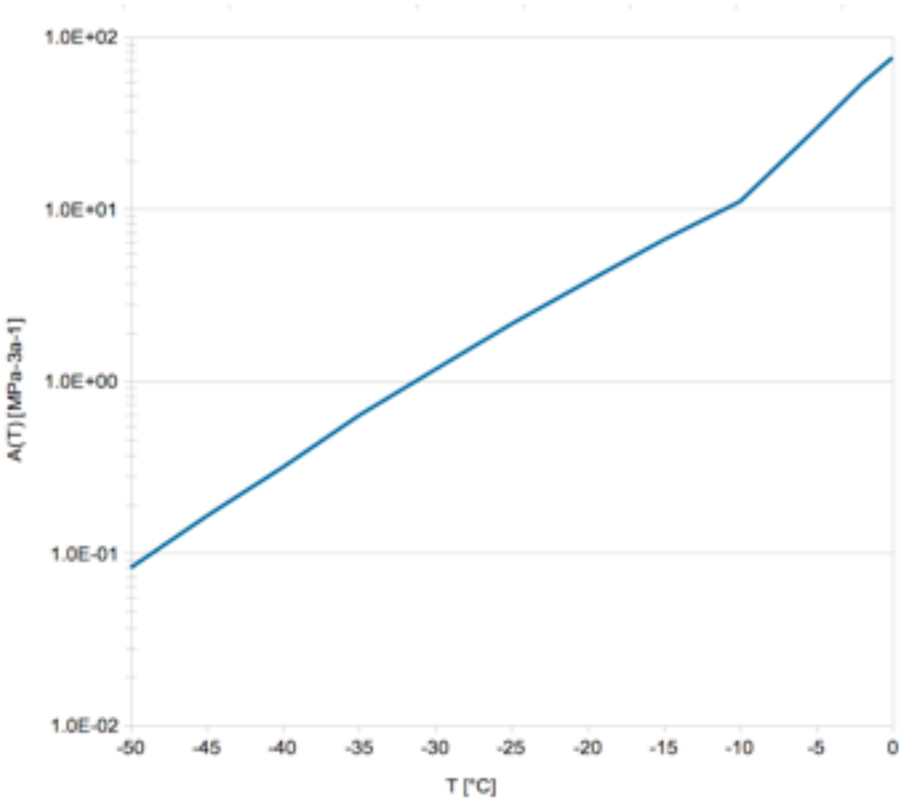
$$A(T') = A(T_0)\exp^{\frac{Q}{R}\left(\frac{1}{T_0}-\frac{1}{T'}\right)}$$

$T' = T - T_m$, with $T_m = 273.15 + 9.8 \times 10^{-8} p_i$ (Clausius-Clapeyron)
Q activation energy
$R = 8.314$ universal gaz constant

Isotropic Ice (Glen's law)

Arrhenius law for temperature dependency

$$A(T') = A(T_0)\exp^{\frac{Q}{R}\left(\frac{1}{T_0}-\frac{1}{T'}\right)}$$

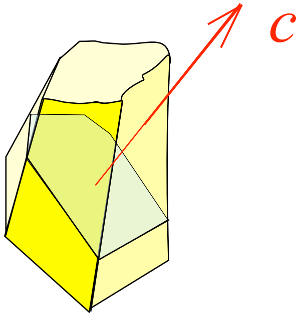


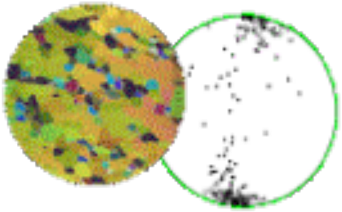
T°C	A [$\mathrm{Pa^{-3}\,s^{-1}}$]	A [$\mathrm{MPa^{-3}\,a^{-1}}$]
0	2.40E-024	7.574E+01
-2	1.70E-024	5.365E+01
-5	9.30E-025	2.935E+01
-10	3.50E-025	1.105E+01
-15	2.10E-025	6.627E+00
-20	1.20E-025	3.787E+00
-25	6.80E-026	2.146E+00
-30	3.70E-026	1.168E+00
-35	2.00E-026	6.312E-01
-40	1.00E-026	3.156E-01
-45	5.20E-027	1.641E-01
-50	2.60E-027	8.205E-02

Recommended values by Cuffey and Patterson [2010]

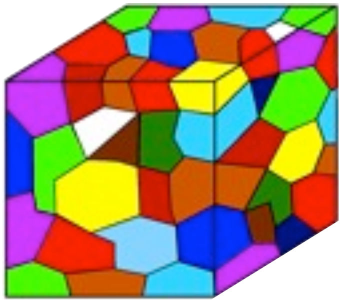
Description of the fabric

Ice crystal





Fabric



Polycrystalline ice

(Castelnau et al., 1996)

- Discrete: two many variables
- Orientation tensors: Yes!

$$\boxed{\begin{aligned} \boldsymbol{a}^{(2)} &= < \boldsymbol{c} \otimes \boldsymbol{c} > \\ \boldsymbol{a}^{(4)} &= < \boldsymbol{c} \otimes \boldsymbol{c} \otimes \boldsymbol{c} \otimes \boldsymbol{c} > \end{aligned}} \longrightarrow \begin{cases} a_1^{(2)} + a_2^{(2)} + a_3^{(2)} = 1 \\ a_1^{(2)},\ a_2^{(2)},\ {}^o\boldsymbol{e}_1,\ {}^o\boldsymbol{e}_2, {}^o\boldsymbol{e}_3 \end{cases}$$

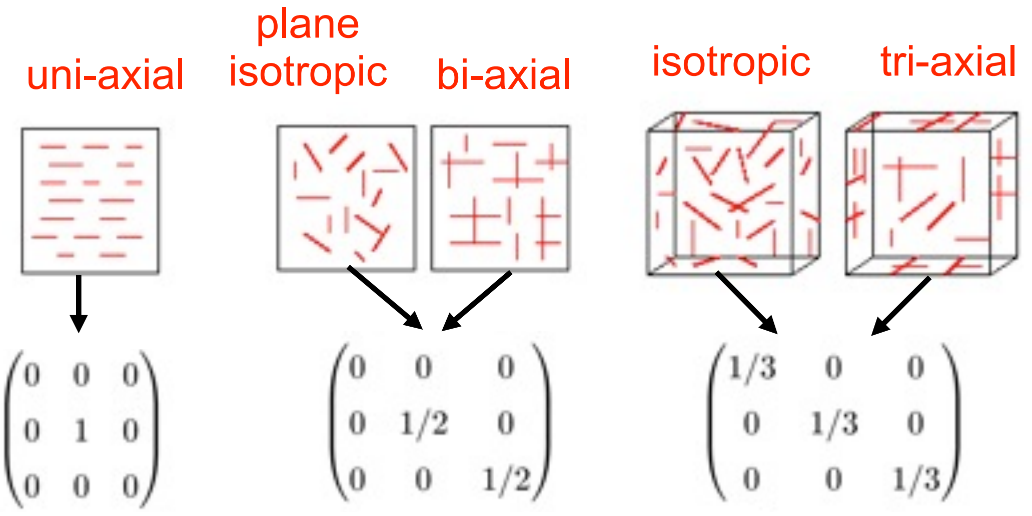
Only 5 variables needed!
(The 2 eigenvalues and 3 eigenvectors of the second order orientation tensor)

Examples of fabric

Orientation tensors

$$\boldsymbol{a}^{(2)} = <\boldsymbol{c} \otimes \boldsymbol{c}>$$

$$\boldsymbol{a}^{(4)} = <\boldsymbol{c} \otimes \boldsymbol{c} \otimes \boldsymbol{c} \otimes \boldsymbol{c}>$$

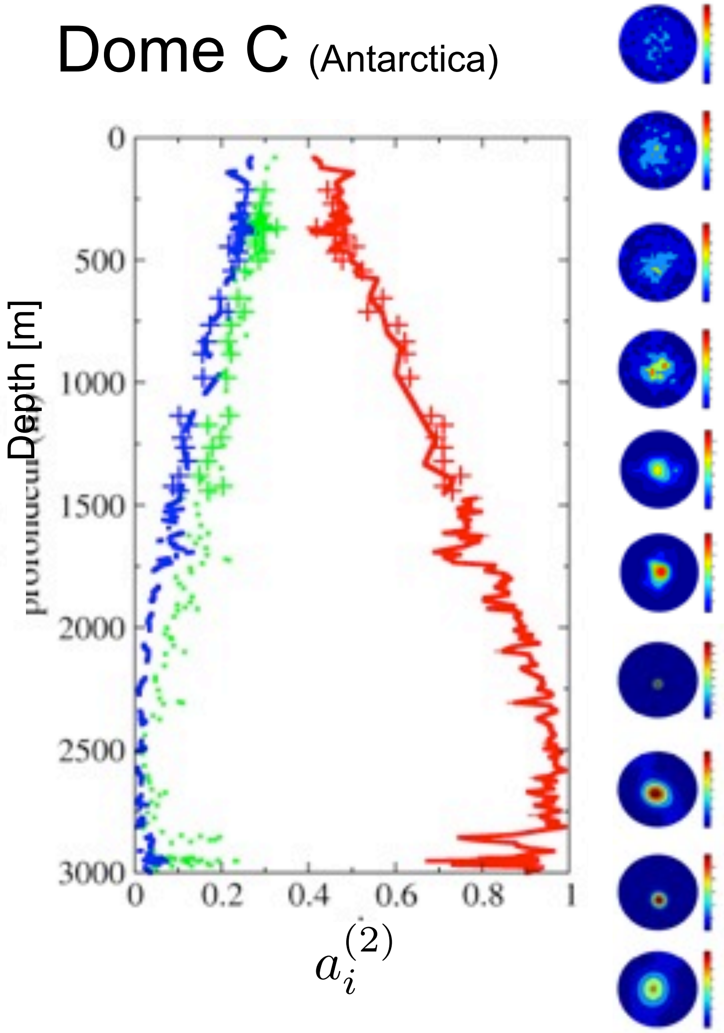


$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \quad \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \end{pmatrix}$$

☹ The knowledge of $\boldsymbol{a}^{(2)}$ is not sufficient

☺ $\boldsymbol{a}^{(2)}$ and $\boldsymbol{a}^{(4)}$ are sufficient

Dome C (Antarctica)



(Wang et al, 2003 ; Durand et al., 2007)

Anisotropic Ice

Two dedicated laws in Elmer/Ice:

Name	GOLF	CAFFE
Anisotropy	Orthotropic	Enhancement factor
Collinear	No	Yes
Calibration	Tabulated using a micro-macro model	From experimental results
Easiness	dedicated solver (AIFlow)	Navier-Stokes Solver + User Function

GOLF: General Orthotropic Flow Law [Gillet-Chaulet et al., 2005, 2006 ; Durand et al., 2009 ; Ma et al., 2010]
CAFFE: Continuum-mechanical, Anisotropic Flow model based on an anisotropic Flow Enhancement factor [Placidi and Hutter, 2006 ; Seddik et al., 2008, 2009 ; Placidi et al., 2010]

Anisotropic Ice

GOLF:

$$\sum_{r=1}^{3}\left[\eta_r tr(\boldsymbol{M}_r\cdot\boldsymbol{D})\boldsymbol{M}_r^D+\eta_{r+3}(\boldsymbol{D}\cdot\boldsymbol{M}_r+\boldsymbol{M}_r\cdot\boldsymbol{D})^D\right]=2A\tau_e^{n-1}\boldsymbol{\tau}$$

$\eta_r=\eta_r(\boldsymbol{a}^{(2)})$, 6 relative viscosities function of the fabric
$\boldsymbol{M}_r={}^o\vec{e}_r\otimes{}^o\vec{e}_r$, 3 structure tensors from the 3 principal axes

CAFFE:

$$D=2EA\tau_e^{n-1}\boldsymbol{\tau}$$

$E=E(\boldsymbol{a}^{(2)})$, 1 scalar enhancement factor function of the fabric

Fabric evolution

For both laws, need an equation describing the fabric evolution, *i.e.* the evolution of $\boldsymbol{a}^{(2)}$

$$\frac{\partial \boldsymbol{a}^{(2)}}{\partial t} + g(\boldsymbol{S}, \boldsymbol{D}, \boldsymbol{a}^{(2)}, \boldsymbol{a}^{(4)}) + \kappa(\boldsymbol{I} - \boldsymbol{a}^{(2)}) = 0$$

recrystallization

Need a closure approximation

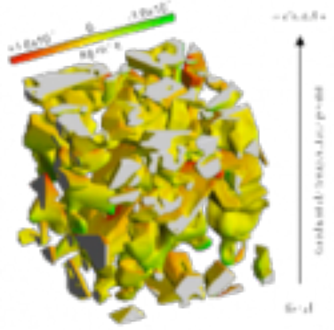
$$\boldsymbol{a}^{(4)} = f(\boldsymbol{a}^{(2)})$$

☺ Only Macroscopic quantities

Gödert, 2003

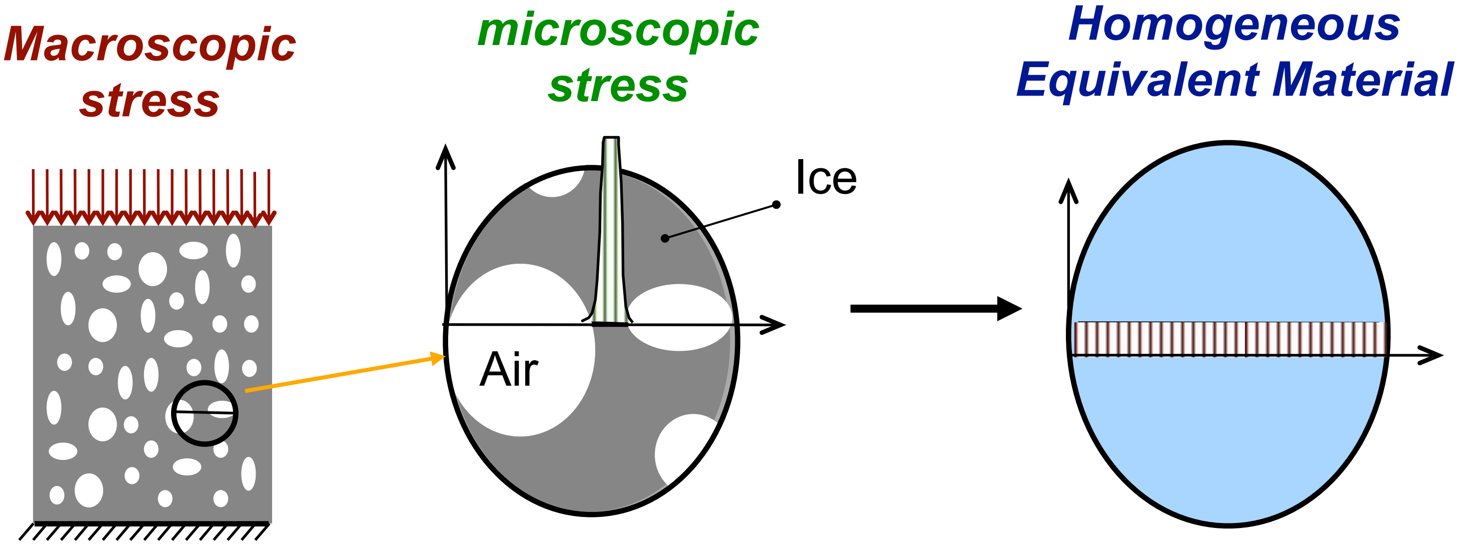
[Gödert, 2003 ; Gillet-Chaulet et al., 2006]

Rheology of snow/firn



Snow/firn = Ice + Air

- Compressible
- Viscosity function of the density



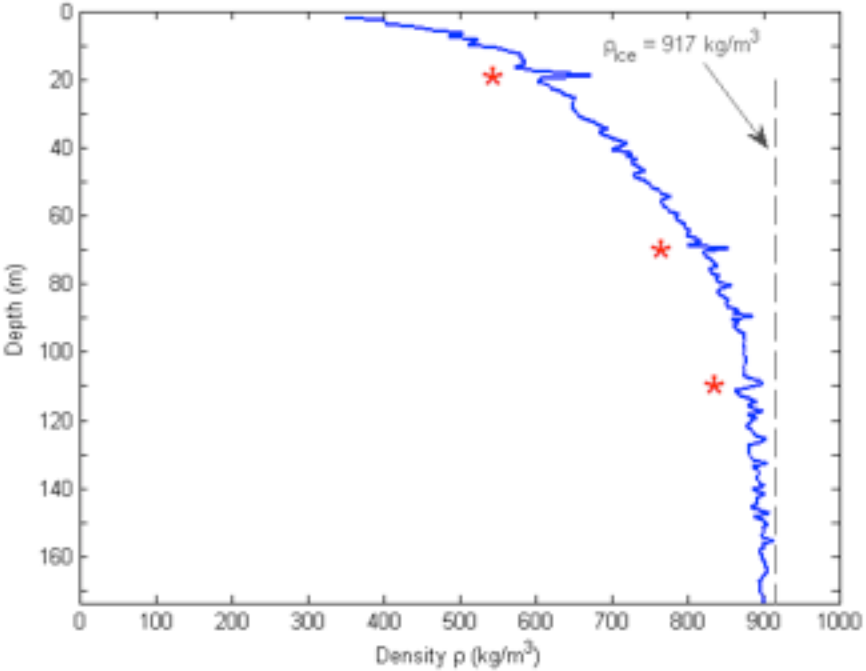
Macroscopic stress

microscopic stress

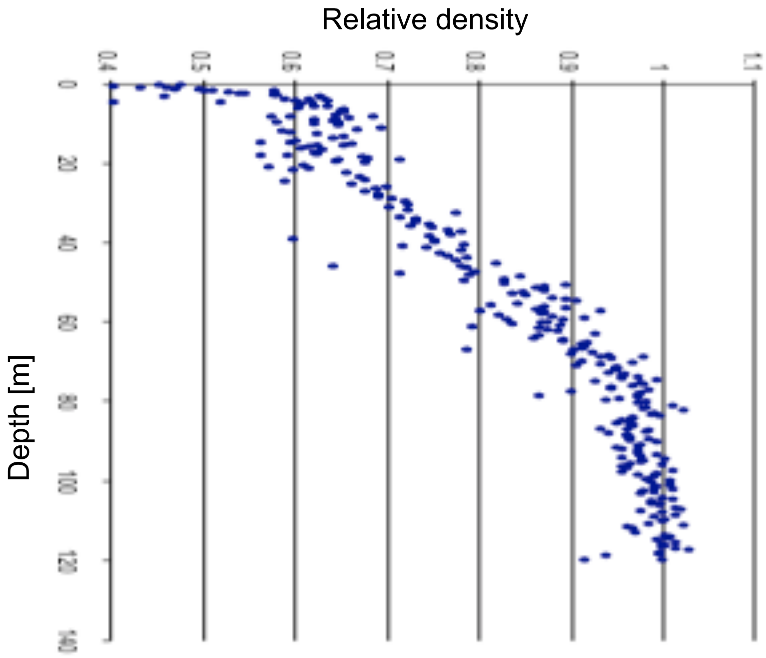
Homogeneous Equivalent Material

[Duva and Crow, 1994]

Observation of density



DYE-3, Greenland (From Niels Bohr Institute)



Dome du Gouter, French Alps

Rheology of snow/firn

Stokes compressible: velocities, isotropic pressure

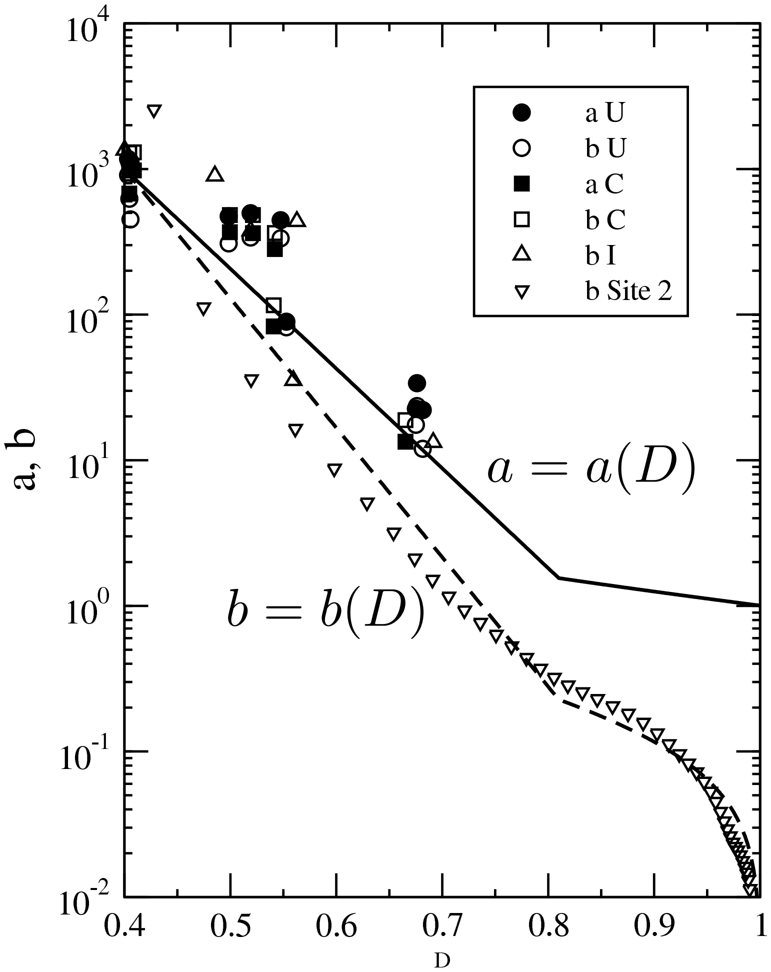
$$\begin{cases} \operatorname{div} \boldsymbol{\sigma} + \rho \boldsymbol{g} = 0 \\ \dfrac{\mathrm{d}\,\rho}{\mathrm{d}\,t} + \operatorname{div} \rho \boldsymbol{u} = 0 \end{cases}$$

Relative density: $D = \rho/\rho_{\mathrm{i}}$

Snow/firn law:

$$\begin{cases} s_{ij} = \dfrac{2}{a} B^{-1/n} \dot{\epsilon}_D^{(1-n)/n} \dot{e}_{ij} \\ p = \dfrac{1}{b} B^{-1/n} \dot{\epsilon}_D^{(1-n)/n} \dot{\epsilon}_{kk} \end{cases}$$

with $\dot{e}_{ij} = \dot{\epsilon}_{ij} - \dfrac{\dot{\epsilon}_{kk}}{3} \delta_{ij}$



[Gagliardini and Meyssonnier, 1997]

Damaged ice : a continuum mechanic approach

We define an effective deviatoric part of the Cauchy stress tensor as:

$$\tilde{\mathbf{S}} = \frac{\mathbf{S}}{(1-D)}$$

Strain is affected only by this effective stress:

$$\tilde{\mathbf{S}} = (A)^{-1/n} \mathbf{I}_{\dot{\varepsilon}_2}^{(1-n)/n} \dot{\boldsymbol{\varepsilon}}. \qquad \mathbf{S} = (A)^{-1/n} (1-D) \mathbf{I}_{\dot{\varepsilon}_2}^{(1-n)/n} \dot{\boldsymbol{\varepsilon}}.$$

By identification with Glen's law, the enhencement factor is a function of damage:

$$E = \frac{1}{(1-D)^n}.$$

Damage is a property of the material at the mesoscale. It is therefore advected by the ice flow, and evolves over time depending on the stress field:

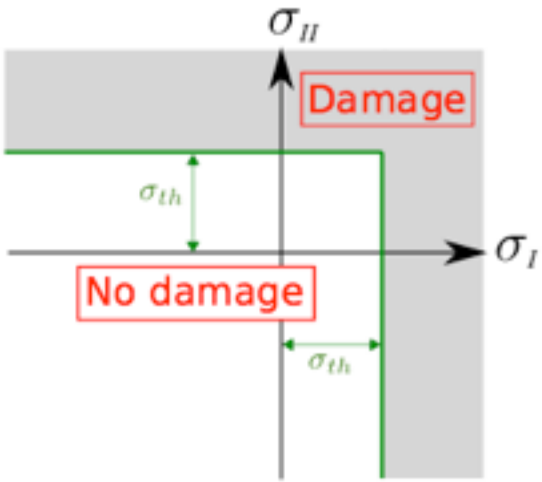
$$\frac{\partial D}{\partial t} + \mathbf{u}\nabla D = \begin{cases} f(\chi) & \text{si } f(\chi) > 0 \\ 0 & \text{sinon} \end{cases}$$

The right-hand side represents a damage source term that can be written as a function of a damage enhencement factor and a damage criterion:

$$f(\chi) = B \cdot \chi(\tilde{\sigma}, \sigma_{th}, D)$$

Damaged ice : a continuum mechanic approach

Here, to describe crevasse opening under pure tension, we use a pure-tensile criterion, described as a function of the maximum principal Cauchy stress:



$$\chi\left(\sigma_{\mathrm{I}},\, \sigma_{\mathrm{th}},\, D\right) = \max\left\{0,\ \frac{\sigma_{\mathrm{I}}}{(1-D)} - \sigma_{\mathrm{th}}\right\}$$

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Glen’s law and Elmer

$$D_{ij} = A\tau_e^{n-1} S_{ij} \quad ; \quad S_{ij} = A^{-1/n} I_{D_2}^{(1-n)/n} D_{ij}$$

$$A(T^{'}) = A(T_0)\exp\frac{Q}{R}\left(\frac{1}{T_0}-\frac{1}{T^{'}}\right)$$

Build-in Glen’s Flaw Low:

```
Material 1
! Glen's flow law
  Viscosity Model = String "Glen"
  Viscosity = Real -9999 ! To avoid warning output
  Glen Exponent = Real 3.0
  Critical Shear Rate = Real 1.0e-10
  ! gives a fixed value in MPa^-3a^-1
  Set Arrhenius Factor = Logical True
  Arrhenius Factor = Real $1.0E-16 * 1.0E18
  Glen Enhancement Factor = Real 1.0
End
```

```
Material 1
  Viscosity Model = String "Glen"
! Viscosity has to be set to a dummy value
! to avoid warning output from Elmer
  Viscosity = Real -9999
  Glen Exponent = Real 3.0
  Critical Shear Rate = Real 1.0e-10
! Rate factors (Paterson value in MPa^-3a^-1)
  Rate Factor 1 = Real 1.258e13
  Rate Factor 2 = Real 6.046e28
! these are in SI units - no problem, as long as
! the gas constant also is
  Activation Energy 1 = Real 60e3
  Activation Energy 2 = Real 139e3
  Glen Enhancement Factor = Real 1.0
! the variable taken to evaluate the Arrhenius law
! in general this should be the temperature relative
! to pressure melting point. The suggestion below plugs
! in the correct value obtained with TemperateIceSolver
  Temperature Field Variable = String "Temp Homologous"
! the temperature to switch between the
! two regimes in the flow law
  Limit Temperature = Real -10.0
! In case there is no temperature variable
  !Constant Temperature = Real -10.0
End
```

Elmer has no restriction on the units system

- Choose the most appropriate units for your silmulations
- Be consistent !!

Anisotropy: related equations

➢ Velocities

Stokes Equations

$$\begin{cases} \operatorname{div} \boldsymbol{\sigma} + \rho \boldsymbol{g} = 0 \\ \operatorname{div} \rho \boldsymbol{u} = 0 \end{cases}$$

\+ behaviour (GOLF, CAFFE)

- Inputs
  - ➢ Fabric
  - ➢ Temperature
  - ➢ Geometry
- Outputs
  - ➢ **Velocities** and isotropic pressure
  - ➢ Stresses, strain-rates

➢ Fabric

$$\frac{\partial \boldsymbol{a}^{(2)}}{\partial t} + g(\boldsymbol{S}, \boldsymbol{D}, \boldsymbol{a}^{(2)}, \boldsymbol{a}^{(4)}) + \kappa(\boldsymbol{I} - \boldsymbol{a}^{(2)}) = 0$$

- Inputs : Velocities, stresses, strain-rates, rotation rate
- BC : Isotropic ice at the surface
- Outputs : **Fabric** field

Fully coupled ϵ_C

➢ Free surface elevation

$$\frac{\partial h}{\partial t} + \frac{\partial h}{\partial x_1} u_1 + \frac{\partial h}{\partial x_2} u_2 - u_3 = b$$

- Inputs Velocities
- Outputs **Surface elevation**

AIFlow Solver

$$\sum_{r=1}^{3}\left[\eta_r tr(\boldsymbol{M}_r\cdot\boldsymbol{D})\boldsymbol{M}_r^D + \eta_{r+3}(\boldsymbol{D}\cdot\boldsymbol{M}_r+\boldsymbol{M}_r\cdot\boldsymbol{D})^D\right] = 2A\tau_e^{n-1}\boldsymbol{\tau}$$

Add the AIFlow solver:

```
Solver 1
  Equation = AIFlow
  Variable = AIFlow
  Variable DOFs = 3                         ! 3 for 2D -- 4 for 3D

  Exported Variable 1 = Temperature         ! Define Temperature Mandatory!!
  Exported Variable 1 DOFS = Integer 1

  Exported Variable 2 = Fabric              ! Define Fabric Variable
  Exported Variable 2 DOFS = Integer 5          ! Mandatory if Isotropic=False

  Exported Variable 3 =  StrainRate         ! Compute SR
  Exported Variable 3 DOFS = Integer 4      ! 4 in 2D  6 in 3D (11,22,33,12,23,31)

  Exported Variable 4 =  DeviatoricStress   ! Compute Stresses
  Exported Variable 4 DOFS = Integer 4      ! 4 in 2D  6 in 3D  (11,22,33,12,23,31)

  Exported Variable 4 =  Spin               ! Compute Spin
  Exported Variable 4 DOFS = Integer 1      ! 1 in 2D  3 in 3D (12,23,31)

  Procedure = "ElmerIceSolvers" "AIFlowSolver_nlS2"
  !Procedure = "ElmerIceSolvers" "AIFlowSolver_nlD2"
End
```

AIFlow Solver

$$\sum_{r=1}^{3}\left[\eta_r tr(\boldsymbol{M}_r \cdot \boldsymbol{D})\boldsymbol{M}_r^D + \eta_{r+3}(\boldsymbol{D} \cdot \boldsymbol{M}_r + \boldsymbol{M}_r \cdot \boldsymbol{D})^D\right] = 2A\tau_e^{n-1}\boldsymbol{\tau}$$

In the Body Force section:

```
  AIFlow Force 2 = Real -0.00899   ! body force, i.e. gravity * density
```

In the Material section:

```
  Powerlaw Exponent = Real 3.0
  Min Second Invariant = Real 1.0e-10   ! Min value for the second invariant of strain-rates
  Reference Temperature = Real -10.0    ! T0 (Celsius)!
  Fluidity Parameter = Real 20.         ! Bn(T0) = 2 x A(T0)
  Limit Temperature = Real -5.0         ! TL  (Celsius)!
  Activation Energy 1 = Real 7.8e04     ! Joule/mol for T<TL
  Activation Energy 2 = Real 7.8e04     ! Joule/mol for T>TL

  Viscosity File = FILE "040010010.Va"    !Contains the tabulated relative viscosities
  Isotropic = Logical False             ! If True, no need of Fabric variable
```

In the Initial Condition section:

```
  Fabric 1 = Real 0.33333333333333 !a2_11
  Fabric 2 = Real 0.33333333333333 !a2_22
  Fabric 3 = Real 0.               !a2_12
  Fabric 4 = Real 0.               !a2_23
  Fabric 5 = Real 0.               !a2_13

  AIFlow 1 = Real 0.0              ! u_1
  AIFlow 2 = Real 0.0              ! u_2
  AIFlow 3 = Real 0.0              ! p for 2D u_3 for 3D
  AIFlow 4 = Real 0.0              ! only for 3D = p
```

AIFlow Solver

$$\sum_{r=1}^{3}\left[\eta_r tr(\boldsymbol{M}_r \cdot \boldsymbol{D})\boldsymbol{M}_r^D + \eta_{r+3}(\boldsymbol{D}\cdot\boldsymbol{M}_r + \boldsymbol{M}_r\cdot\boldsymbol{D})^D\right] = 2A\tau_e^{n-1}\boldsymbol{\tau}$$

In Boundary Condition section:

Dirichlet condition for velocity:

```
AIFlow 1 = Real 0.
AIFlow 2 = Real 0.
```

Neumann condition for AIFlow:

```
Normal force = Real 0.              ! a pressure along the normal of the surface
Force 1 = Real 0.                   ! stress along x (Sxn, with n the surface normal)
Force 2 = Real 0.                   ! stress along y (Syn)
Force 3 = Real 0.                   ! stress along z (Szn)
AIFlow Slip Coeff 1 = Real 0.1      ! Slip coefficient in direction 1
```

Normal-Tangential boundary condition (for Dirichlet and Neumann):

```
Normal-Tangential AIFlow = Logical True
```

CAFFE User Function

$$D = 2EA\tau_e^{n-1}\boldsymbol{\tau}$$
$$E = E(\boldsymbol{a}^{(2)}), \text{ 1 scalar enhancement factor function of the fabric}$$

In Material Section:

```
Viscosity Model = String "power law"
Viscosity Exponent = Real MATC "1.0/3.0"

Viscosity =  Variable Temp
        Real Procedure "./CaffeFlow" "caffeGetViscosity"

 Activation Energies (2) = Real 6.0E04 1.39E05
 Arrhenius Factors (2) = Real 3.985E-13  1.916E03
 Enhancement Factor = Real 1.0
 Limit Temperature = Real -10.0 ! switching between the two values
 Temp Upper Limit = Variable Pressure
        Real Procedure  "IceFlowProperties" "getPressureMeltingPoint"

 Anisotropic Enhancement factor = Real 10.0
 Critical Enhancement factor = Real 0.0001
```

Contact: Hakime Seddik (hakime@pop.lowtem.hokudai.ac.jp)

Fabric Solver

$$\frac{\partial \boldsymbol{a}^{(2)}}{\partial t} + g(\boldsymbol{S}, \boldsymbol{D}, \boldsymbol{a}^{(2)}, \boldsymbol{a}^{(4)}) + \kappa(\boldsymbol{I} - \boldsymbol{a}^{(2)}) = 0$$

Add the Fabric solver:

```
Solver 2
  Equation = Fabric
  Variable = -nooutput Compfab      ! dumy variable (Fabric variable exported from AIFlow)
  Variable DOFs = 1                 ! FabricSolver compute each variable independently, Picard Type iterations

  Procedure = "ElmerIceSolvers" "FabricSolver"
  Discontinuous Galerkin = Logical True
End
```

In the Material section:

```
 Interaction Parameter = Real 0.    ! 0 => Fabric Evolution function of Strain-rates
                                    ! 1=> Fabric Evolution function of dev stresses
                                    ! If not defined set to the default value given in Viscosity File
 Diffusion Parameter = Real 0.      ! Diffusion term. By default set to 0 if not defined
```

In the Boundary Condition section:

Only Dirichlet BC for Fabric (required for inflow boundary condition, no condition for outflow)

```
 Fabric 1 = Real $1./3.     !a2_11
 Fabric 2 = Real $1./3.     !a2_22
 Fabric 3 = Real 0.          !a2_12
 Fabric 4 = Real 0.         !a2_23
 Fabric 5 = Real 0.          !a2_13
```

Here, isotropic fabric (as for the upper surface)

Anisotropy in Elmer/Ice references

Sun, B., Moore, J. C., Zwinger, T., Zhao, L., Steinhage, D., Tang, X., Zhang, D., Cui, X., and Martín, C., 2014. ***How old is the ice beneath Dome A, Antarctica?***, The Cryosphere, 8, 1121-1128, doi:10.5194/tc-8-1121-2014.

Zwinger, T., M. Schäfer, C. Martín, and J.C. Moore, 2014. ***Influence of anisotropy on velocity and age distribution at Scharffenbergbotnen blue ice area***, The Cryosphere, **8**, 607-621, doi:10.5194/tc-8-607-2014.

Martín, C., G.H. Gudmundsson and E.C. King 2014. ***Modelling of Kealey Ice Rise, Antarctica, reveals stable ice-flow conditions in East Ellsworth Land over millennia***, J. Glaciol., **60**, 139-146, doi:10.3189/2014JoG13J089

Martín, C. and G. H. Gudmundsson, 2012. ***Effects of nonlinear rheology, temperature and anisotropy on the relationship between age and depth at ice divides***, *The Cryosphere*, **6**, 1221-1229, doi:10.5194/tc-6-1221-2012.

Seddik H., R. Greve, T. Zwinger and L. Placidi, 2011. ***A full-Stokes ice flow model for the vicinity of Dome Fuji, Antarctica, with induced anisotropy and fabric evolution***, The Cryosphere, **5**, 495-508, doi:10.5194/tc-5-495-2011.

Gillet-Chaulet, F., R.C.A. Hindmarsh, H.F.J. Corr, E.C. King, and A. Jenkins, 2011. ***In-situ quantification of ice rheology and direct measurement of the Raymond Effect at Summit, Greenland using a phase-sensitive radar***, Geophys. Res. Lett., **38**, L24503, doi: 10.1029/2011GL049843

Ma Y., O. Gagliardini, C. Ritz, F. Gillet-Chaulet, G. Durand and M. Montagnat, 2010. ***Enhancement factors for grounded ice and ice shelves inferred from an anisotropic ice-flow model***, J. Glaciol., **56**(199), 805-812.

Durand G., F. Gillet-Chaulet, A. Svensson, O. Gagliardini, S. Kipfstuhl, J. Meyssonnier, F. Parrenin, P. Duval and D. Dahl-Jensen, 2007. ***Change of the ice rheology with climatic transitions – implication on ice flow modelling and dating of the EPICA Dome C core***, Clim. Past., 3, 155-167

Gillet-Chaulet F., O. Gagliardini, J. Meyssonnier, T. Zwinger, J. Ruokolainen, 2006. ***Flow-induced anisotropy in polar ice and related ice-sheet flow modelling***, J. Non-Newtonian Fluid Mech. **134**, p. 33-43

Snow/firn: related equations

➢ Velocities

Stokes Equations

$$\begin{cases} \operatorname{div} \boldsymbol{\sigma} + \rho \boldsymbol{g} = 0 \\ \operatorname{div} \rho \boldsymbol{u} = 0 \end{cases}$$

+ behaviour (Snow/Firn)

- Inputs
  - ➢ Density
  - ➢ Temperature
  - ➢ Geometry
- Outputs
  - ➢ **Velocities** and isotropic pressure
  - ➢ Stresses, strain-rates

➢ Density

$$\frac{\mathrm{d}\,\rho}{\mathrm{d}\,t} + \operatorname{div} \rho \boldsymbol{u} = 0$$

- Inputs : Velocities, stresses, strain-rates, rotation rate
- BC : snow density at the upper surface
- Outputs : **Density** field

Fully coupled ϵ_C

➢ Free surface elevation

$$\frac{\partial h}{\partial t} + \frac{\partial h}{\partial x_1} u_1 + \frac{\partial h}{\partial x_2} u_2 - u_3 = b$$

- Inputs Velocities
- Outputs **Surface elevation**

Porous Solver

```
! this is the compressible Stokes solver
!----------------------------------------
Solver 1
  Equation = String "PorousFlow"
  Procedure = "ElmerIceSolvers" "PorousSolver"
  Variable = "Porous"
  Variable DOFs = 4 ! 4 in 3D (u,v,w,p) ; 3 in 2D (u,v,p)

  Optimize Bandwidth = False
! Use p elements
! Element = "p:1 b:4"
! Stablization Method = String pBubbles

 ! Exported Variable 1 = String "Relative Density"
 ! Exported variable 1 DOFs = Integer 1

! switch that in for post-processing issues only
   Exported Variable 2 = String "StrainRate"
   Exported variable 2 DOFs = Integer 6 ! 4 in 2D, 6 in 3D
   Exported Variable 3 = String "DeviatoricStress"
   Exported variable 3 DOFs = Integer 6 ! 4 in 2D, 6 in 3D
   Exported Variable 4 = String "Spin"
   Exported variable 4 DOFs = Integer 3 ! 1 in 2D, 3 in 3D

  Linear System Solver = 'Direct'
! Only Picard linearization available for this solver
  Nonlinear System Convergence Tolerance = 1.0E-05
  Nonlinear System Max Iterations = 50

  Steady State Convergence Tolerance = 1.0E-03
End
```

```
! Gravity force
Body Force 1
  Porous Force 1 = Real 0.0E00
  Porous Force 2 = Real 0.0E00
  Porous Force 3 = Real $gravity*rhoi
End
```

```
Material 1
  Powerlaw Exponent = Real $n
  Min Second Invariant = Real 1.0E-10
  Fluidity Parameter = Real $B  ! MPa^{-3}a^{-1}
! Density as a function of relative density
  Density = Variable Relative Density
        Real MATC "tx*rhoi"
End
```

```
! Neumann type boundary condition
Boundary Condition 1
    Force 3 = Real -0.01
End
! or
Boundary Condition 1
  Normal Force = Real -0.01
End

! Dirichlet / Newton Boundary condition
! here: zero normal velocity and sliding
Boundary Condition 2
  Target Boundaries  = 2
  Normal-tangential Porous = True
  Porous 1 = Real 0.0
  Porous Slip Coeff 2 = Real 0.1
  Porous Slip Coeff 3 = Real 0.1
End
```

Density Solver

$$\frac{\mathrm{d}\,\rho}{\mathrm{d}\,t} + \operatorname{div}\,\rho\boldsymbol{u} = 0$$

This is a generic Advection-Reaction equation

=> Use the Elmer AdvectionReaction Solver

```
Solver 8
  Equation = "AdvReact"
  Exec Solver = "After Timestep"
  Procedure = File "AdvectionReaction" "AdvectionReactionSolver"
  ! this is the DG variable, which is not part of the output
  Variable =  -nooutput "DGdens"
  ! this tells that the solver is run on DG mesh
  Discontinuous Galerkin = Logical True
  ! the solver can account for upper and lower limits of the variable
  ! imposed by formulation of an variational inequality (VI)
  ! next line switches the VI to be accounted for
  Limit Solution = Logical True

  Linear System Solver = Iterative
  Linear System Iterative Method = BiCGStab
  Linear System Max Iterations  = 1000
  Linear System Preconditioning = ILU1
  Linear System Convergence Tolerance = 1.0e-06
  ! Variational inequality makes it a non-linear problem
  Nonlinear System Max Iterations = 40
  Nonlinear System Min Iterations = 2
  Nonlinear System Convergence Tolerance = 1.0e-04

  ! This is the variable that is used to interpolate
  ! the DG solution to the regular FEM mesh in order
  ! to get a correct output
  Exported Variable 1 = Relative Density
  Exported Variable 1 DOFS = 1
End
```

```
Body Force 1
 ...
  DGDens Source = Real 0.0
End
```

```
Material 1
 ..
 ! Relative density must stay < 1
 DGDens Upper Limit = Real 1.0

 ! a minimum relative density is recommended for the Porous solver
 DGDens Lower Limit = Real 0.3

 !Reaction rate is equal to zero
 DGDens Gamma = Real 0.0
End
```

```
Initial Condition 1
  ...
  DGDens = Real 0.4
End

! only Dirichlet BC can be set
! the solver automatically uses this
! condition only on inflow boundaries
! outflow boundaries are ignored
Boundary Condition 2
  Name = "surf"
  Target Boundaries = 2
  Body ID = 2
  ...
  ! relative density on the upper surface
  DGDens = Real 0.4
End
```

Porous solver in Elmer/Ice references

The snow/firn rheological law is from:
Gagliardini O. and J. Meyssonnier, 1997. Flow simulation of a firn covered cold glacier. Annals of Glaciol., 24, p. 242-248.

Its implementation within Elmer/Ice and an application are presented in this reference:
Zwinger T. , R. Greve, O. Gagliardini , T. Shiraiwa and M. Lyly, 2007. A full Stokes-flow thermo-mechanical model for firn and ice applied to the Gorshkov crater glacier, Kamchatka. Annals of Glaciol., 45, p. 29-37.

Applications:

Gilbert A., C. Vincent, O. Gagliardini, J. Krug and E. Berthier, 2015. ***Assessment of thermal change in cold avalanching glaciers in relation to climate warming***, Geophys. Res. Lett., **42**, doi:10.1002/2015GL064838.

Gilbert, A., O. Gagliardini, C. Vincent, and P. Wagnon, 2014. ***A 3-D thermal regime model suitable for cold accumulation zones of polythermal mountain glaciers***, J. Geophys. Res. Earth Surf., **119**, doi:10.1002/2014JF003199

Damage: related equations

- Velocities

Stokes Equations

$$\begin{cases} \operatorname{div} \boldsymbol{\sigma} + \rho \boldsymbol{g} = 0 \\ \operatorname{div} \rho \boldsymbol{u} = 0 \end{cases}$$

+ Glen's law for damaged ice

- Inputs
  - Damage
  - Temperature
  - Geometry
- Outputs
  - **Velocities** and isotropic pressure
  - Stresses, strain-rates

- Damage

$$\frac{\partial D}{\partial t} + \mathbf{u} \nabla D = \begin{cases} f(\chi) & \text{si } f(\chi) > 0 \\ 0 & \text{sinon} \end{cases}$$

- Inputs : Velocities, Deviatoric stresses
- BC : Damage at inflow boundaries
- Outputs Damage field

Fully coupled ϵ_C

- Free surface elevation

$$\frac{\partial h}{\partial t} + \frac{\partial h}{\partial x_1} u_1 + \frac{\partial h}{\partial x_2} u_2 - u_3 = b$$

- Inputs Velocities
- Outputs **Surface elevation**

Damage Solver

$$\frac{\partial D}{\partial t} + \mathbf{u}\nabla D = \begin{cases} f(\chi) & \text{si } f(\chi) > 0 \\ 0 & \text{sinon} \end{cases}$$

$$f(\chi) = B \cdot \chi(\tilde{\sigma}, \sigma_{th}, D)$$

This is a generic Advection-Reaction equation

=> Use the Elmer AdvectionReaction Solver

```
Solver 3
  Equation = Sij
  Procedure = "ElmerIceSolvers" "ComputeDevStress"
[...]
  Exported Variable 1 = Stress[Sxx:1 Syy:1 Szz:1 Sxy:1 Syz:1 Sxz:1]
  Exported Variable 1 DOFs = 6
!Additionnally, for output visualisation, the damage criterion Chi is saved as a variable named Chi, which !
need to be exported in a solver, such as :
  Exported Variable 2 = -dofs 1 "Chi"
[...]
End

Solver 8
  Equation = "AdvReact"
  Exec Solver = "After Timestep"
  Procedure = File "AdvectionReaction" "AdvectionReactionSolver"
  ! this is the DG variable, which is not part of the output
  Variable =  -nooutput "DGdamage"

  ! this tells that the solver is run on DG mesh
  Discontinuous Galerkin = Logical True
  ! the solver can account for upper and lower limits of the variable
  ! imposed by formulation of an variational inequality (VI)
  ! next line switches the VI to be accounted for
  Limit Solution = Logical True


  ! This is the variable that is used to interpolate
  ! the DG solution to the regular FEM mesh in order
  ! to get a correct output
  Exported Variable 1 = Damage
End
```

```
Body Force 1
  DGD Source = Variable Damage
    Real Procedure "ElmerIceUSF" "SourceDamage"
End
```

```
Material 1
 ..
 Glen Enhancement Factor = Variable Damage
    Real Procedure "ElmerIceUSF" "EnhancementFactor"

  Damage Enhancement Factor = Real 2.00 ! damage enhancement factor
  Damage Parameter sigmath = Real 0.05 ! stress threshold for damage
increase
End
```

Coupling with Stokes and Glen's flow law:

$$\boxed{E = \frac{1}{(1-D)^n}}$$

Damage solver in Elmer/Ice references

Krug, J., G. Durand, O. Gagliardini and J. Weiss, 2015. ***Modelling the impact of submarine frontal melting and ice mélange on glacier dynamics***, The Cryosphere, **9**, 989-1003, doi:10.5194/tc-9-989-2015.

Krug, J., J. Weiss, O. Gagliardini and G. Durand, 2014. ***Combining damage and fracture mechanics to model calving***, The Cryosphere, **8**, 2101-2117, doi:10.5194/tc-8-2101-2014.