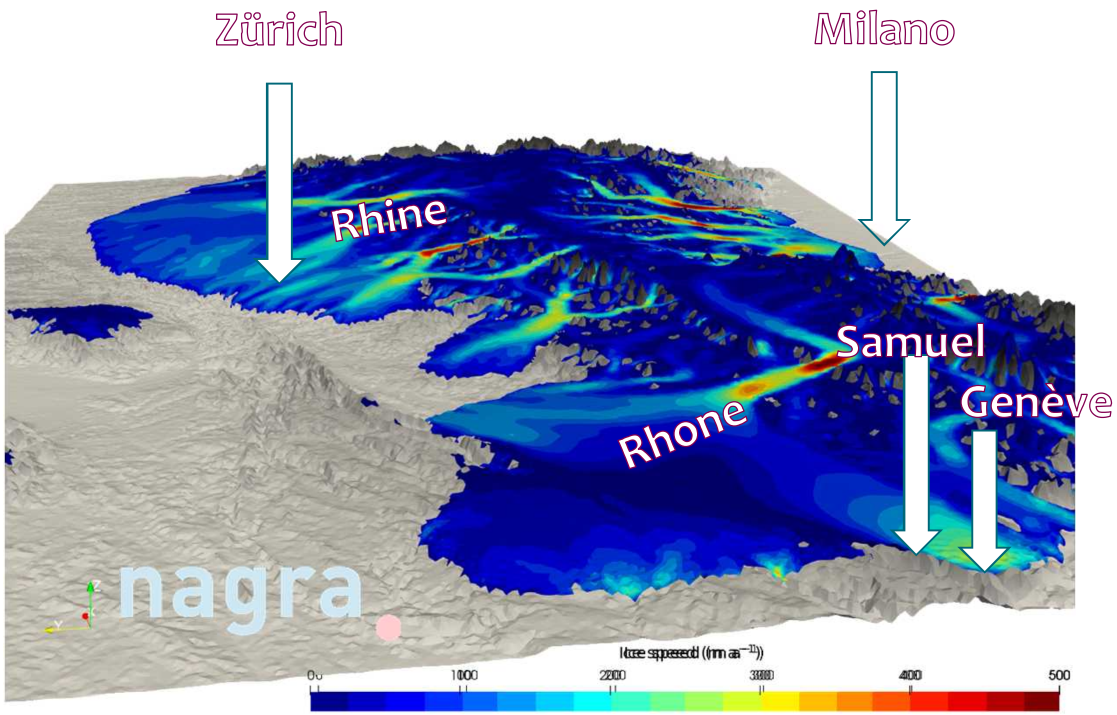
Stabilization scheme of free surface problems

André Löfgren, Josefin Ahlkrona,

Christian Helanow and Thomas Zwinger

Free surface stability on rugged terrain

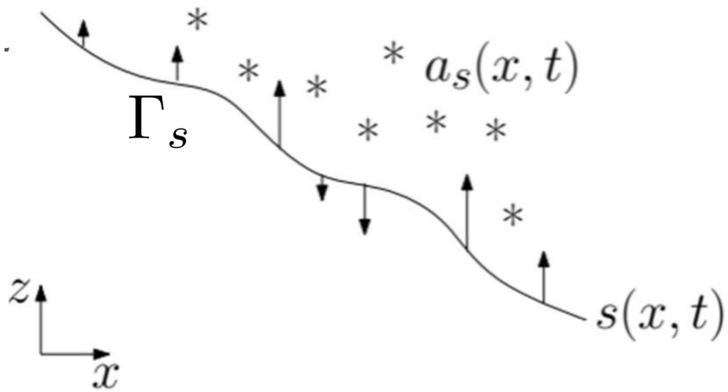


Simulation
D. Cohen
for NAGRA

The free-surface equation

- Surface evolution governed by the kinematic free-surface equation

$$\frac{\partial s}{\partial t} + u_x^s \frac{\partial s}{\partial x} + u_y^s \frac{\partial s}{\partial y} = u_z^s + a_s$$



  - Mass balance between accumulation
  - and the flow of ice.
  - Coupled to the Stoke's flow through the surface velocities.

The Stokes equation

- Bilinear form

$$a((\mathbf{u},p),(\mathbf{v},q)) = (2\eta(\mathbf{u})\dot{\varepsilon}(\mathbf{u}) : \dot{\varepsilon}(\mathbf{v}))_\Omega - ((\nabla\cdot\mathbf{u})q)_\Omega - (p\nabla\cdot\mathbf{v})_\Omega$$

- Linear form

$$L(\mathbf{v}) = (\mathbf{f}\cdot\mathbf{v})_\Omega$$

- Coupled system suffers from instabilities.

The free-surface stabilization algorithm (FSSA)

- Taylor expand force term in Stokes weak form to mimic an implicit time-stepping scheme (Kaus et al. 2010)[1]

$$(\mathbf{f}\cdot\mathbf{v})_{\Omega^{k+1}} \approx (\mathbf{f}\cdot\mathbf{v})_{\Omega^k} + \theta\Delta t((\mathbf{u}\cdot\mathbf{n})(\mathbf{f}\cdot\mathbf{v}))_{\Gamma_s^k}$$

- Adapt to ice-sheet modeling by adding accumulation (Löfgren, Ahlkrona, Helanow 2022)[2]

$$(\mathbf{f}\cdot\mathbf{v})_{\Omega^{k+1}} \approx (\mathbf{f}\cdot\mathbf{v})_{\Omega^k} + \theta\Delta t((\mathbf{u}\cdot\mathbf{n})(\mathbf{f}\cdot\mathbf{v}))_{\Gamma_s^k} + \theta\Delta t((a_s\mathbf{z}\cdot\mathbf{n})(\mathbf{f}\cdot\mathbf{v}))_{\Gamma_s^k}$$

from advection

from accumulation/ablation

Implementation in Elmer

- `IncompressibleNSVec`
- `'full'` is the complete implementation from below
- 2 version originating from initial approximation (now implemented in matrix) declaring a slip coefficient
  - in normal-direction based on vertical velocity (`'normal'`)
  - in vertical-direction (dim) based on normal velocity (`'transposed'`)

```
SELECT CASE(FSSAFlag)
  ! approximation with normal pointing into z-direction
CASE ('normal')
  DO p=1,nd
    DO q=1,nd
      DO i=dim,dim
        STIFF( (p-1)*c+dim,(q-1)*c+i ) = &
             STIFF( (p-1)*c+dim,(q-1)*c+i ) + &
             s * FSSAcoeff * Basis(q) * Basis(p) * Normal(i)
      END DO
    END DO
  END DO
! version 2, transposed
CASE ('transposed')
  DO p=1,nd
    DO q=1,nd
      DO i=1,dim
        STIFF( (p-1)*c+i,(q-1)*c+dim ) = &
             STIFF( (p-1)*c+i,(q-1)*c+dim ) + &
             s * FSSAcoeff * Basis(q) * Basis(p) * Normal(i)
      END DO
    END DO
  END DO
CASE ('full') ! full entry matrix FSSA
  DO p=1,nd
    DO q=1,nd
      DO i=1,dim
        STIFF( (p-1)*c+dim,(q-1)*c+i ) = &
             STIFF( (p-1)*c+dim,(q-1)*c+i ) + &
             s * FSSAcoeff * Basis(q) * Basis(p) * Normal(i)
        !PRINT *, "K(",p,q,i,")=", s * FSSAcoeff * Basis(q) * Basis(p) * Normal(i), STIFF( (p-1)*c+dim,(q-1)*c+i )
      END DO
    END DO
  END DO
```

$$(\mathbf{f}\cdot\mathbf{v})_{\Omega^{k+1}} \approx (\mathbf{f}\cdot\mathbf{v})_{\Omega^k} + \theta\Delta t((\mathbf{u}\cdot\mathbf{n})(\mathbf{f}\cdot\mathbf{v}))_{\Gamma^k}$$

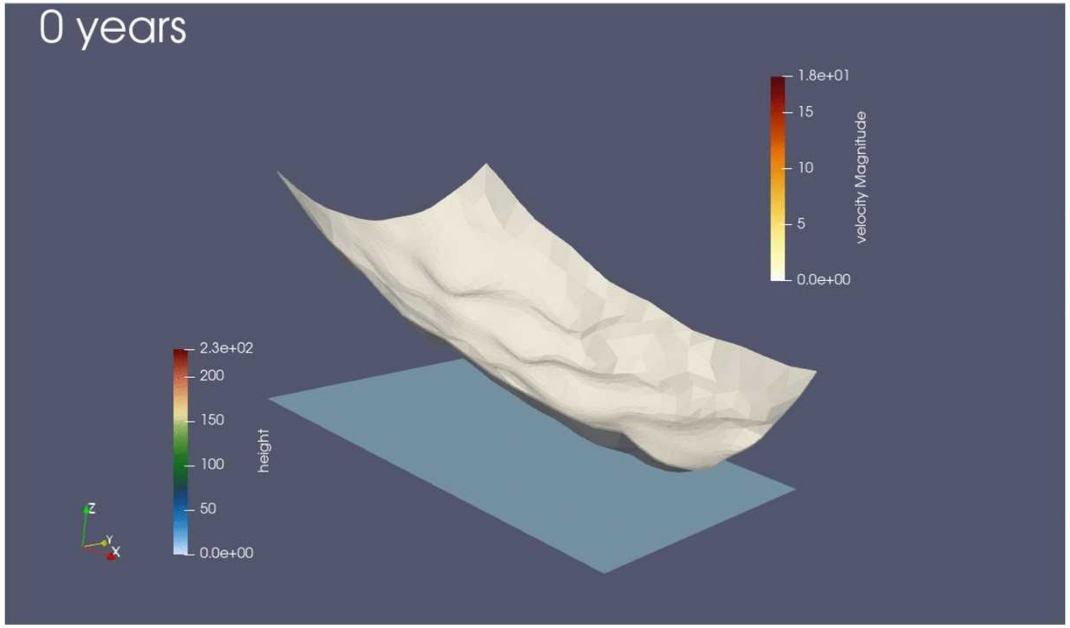
Implementation in Elmer

- In free-surface BC add 2 keywords:
- **`FSSA Coefficient`**:
  - $\Theta \in [0,1]$
- **`FSSA Flag`**:
  - `full`
  - `normal`
  - `transposed`
  - `none` (=default)

```
!---LUA BEGIN
! year_s=365.25*24*3600
! pa_to_mpa=1.0E-6
! rho=910*year_s^(-2.0)*pa_to_mpa
! gravity=9.8*year_s^(2.0)
! time_step=5.0
! theta=1
! stabilization_parameter=theta*time_step*rho*gravity
!---LUA END

Boundary Condition 3
  Name = "surface"
  Top Surface = Equals "Zs"
  Body ID = 2
  FSSA Coefficient = Real #stabilization_parameter
  FSSA Flag = String "full"  ! "normal" "transposed"
End
```

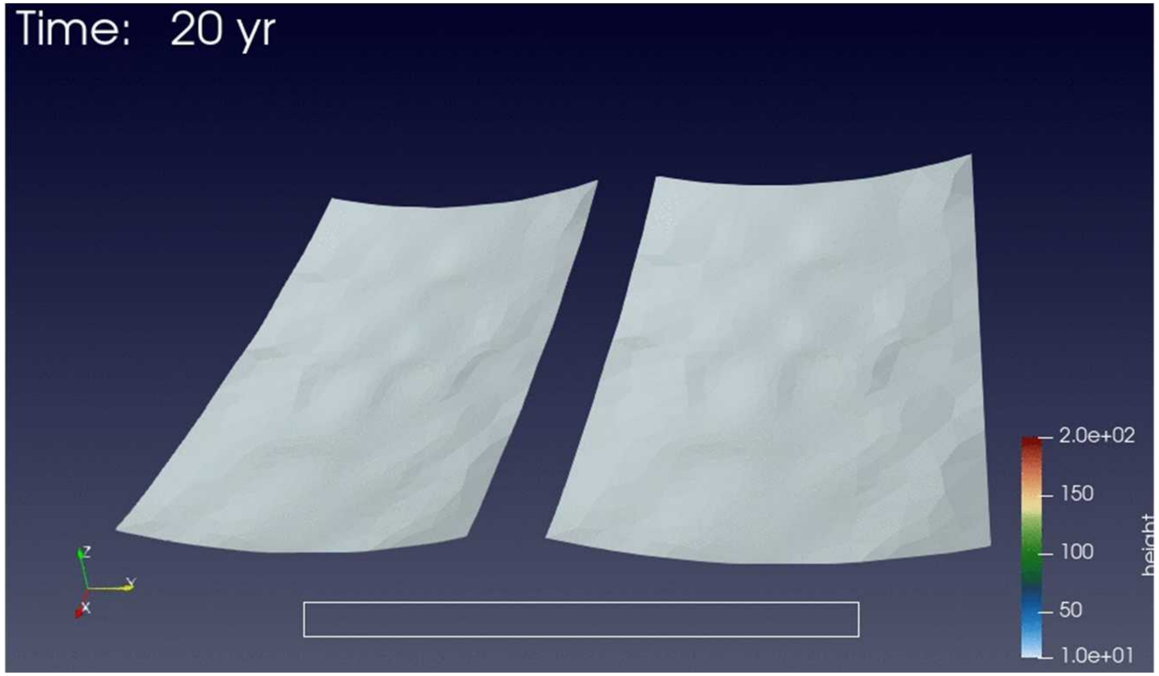
Synthetic test-case



- Synthetic landscape using algorithm also used in gaming (Perlin)
- $8 \times 5 = 40\ \mathrm{km}^2$
- Accumulation in uppermost part, no ablation!
- Low slip in upper part, high slip in lower

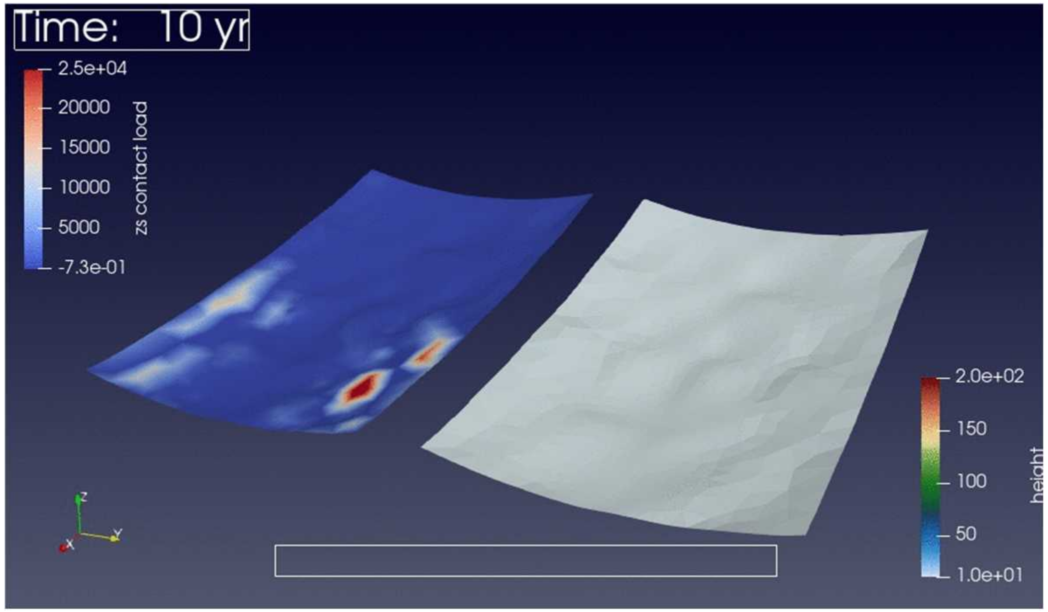
Synthetic test-case

- Comparison between run with full FSSA (left) and no surface stabilization (right)
- Timestep-size 20 years
- Minimum mesh size 100 m
- Maximum front velocities ~20 m/a

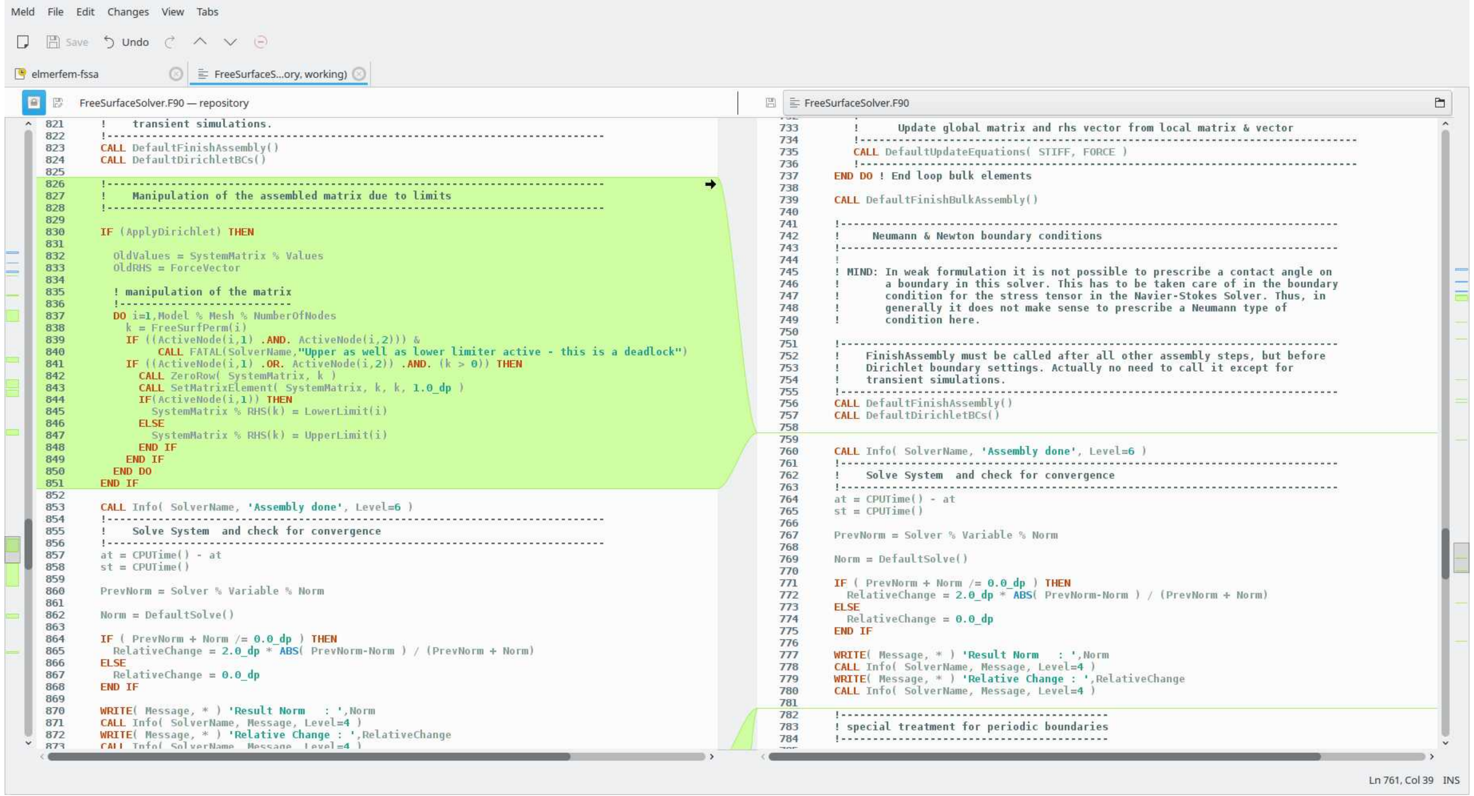


Synthetic test-case

- Minimum flow height 10 m
- Need limiters, particular in steep, non-glaciated parts
- Built-in limiters (`Apply Dirichlet=True`) in `FreeSurfaceSolver` did not work
- Suggest to use library limiters from now on



Changes to `FreeSurfaceSolver`



Changes to `FreeSurfaceSolver`

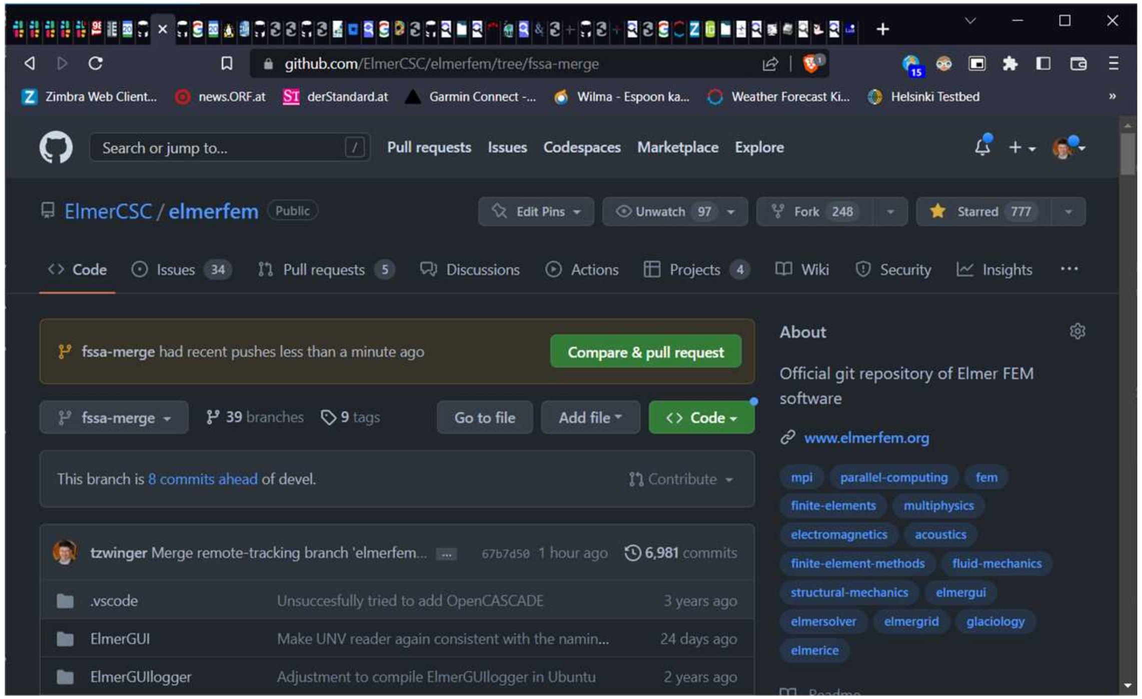
```
> gsu
On branch merge-elmercsc
Your branch is up-to-date with 'origin/merge-elmercsc'.
Changes to be committed:
  (use "git reset HEAD <file>..." to unstage)

        modified:   elmerice/Tests/AIFlowSolve/AIFlow.sif
        modified:   elmerice/Tests/Buoyancy/buoyancy.sif
        modified:   elmerice/Tests/Calving2D/calving2d.sif
        modified:   elmerice/Tests/Calving3D/calving3d.sif
        modified:   elmerice/Tests/Calving3D_lset/calvingMMG.sif
        modified:   elmerice/Tests/Contact/cavity.sif
        modified:   elmerice/Tests/Damage/damage.sif
        modified:   elmerice/Tests/GL_MISMIP/mismip.sif
        modified:   elmerice/Tests/Grounded/grounded.sif
        modified:   elmerice/Tests/MISMIP_FS-SSA/coupled.sif
        modified:   elmerice/Tests/Teterousse3a/teterousse3a.sif
        modified:   elmerice/Tests/Teterousse3a_b/test.sif
        modified:   elmerice/Tests/Teterousse_DeformHeat/test.sif
        modified:   fem/src/modules/FreeSurfaceSolver.F90
        modified:   fem/tests/freesurf_ltd/Free_ltd.sif

Untracked files not listed (use -u option to show untracked files)
> git commit -m "removed built-in Dirichlet limiters in FreeSurfaceSolver and changing
> git commit -m "removed built-in Dirichlet limiters in FreeSurfaceSolver and changing tests accord
e library-provided limiters instead"
[merge-elmercsc c4507f5] removed built-in Dirichlet limiters in FreeSurfaceSolver and changing tests
ly to use library-provided limiters instead
 15 files changed, 102 insertions(+), 241 deletions(-)
```

Conclusions

- New branch under https://github.com/ElmerCSC/elmerfem/tree/fssa-merge
- Introduced FSSA into `IncompressibleNSVec`
  - Not in legacy Navier-Stokes!
- Significant changes to `FreeSurfaceSolver`
  - moving from built-in to library limiters
  - update of test



References

[1] Kaus, B.J.P., et al., *A stabilization algorithm for geodynamic numerical simulations with a free surface*. **Phys. Earth Planet. In.** (2010), doi:10.1016/j.pepi.2010.04.007

[2] Löfgren, A., et al., *Increasing stable time-step sizes of the free-surface problem arising in ice-sheet simulations*. **J. Comput. Phys. X** (2022), doi:10.1016/j.jcpx.2022.100114

CSC

facebook.com/CSCfi

twitter.com/CSCfi

youtube.com/CSCfi

linkedin.com/company/csc---it-center-for-science

github.com/CSCfi

https://github.com/ElmerCSC/elmerfem